Analytic Capacity

and the

Subadditivity Problem

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Eight eights are sixty-four,
          Multiply by seven.
When it's done,
Carry one
          And take away eleven.
Nine nines are eighty-one
          Multiply by three.
If it's more,
Carry four,
          And then it's time for tea.

A.A. Milne

(Now we are Six)

INTRODUCTION AND ABSTRACT

The systematic study of analytic capacity started in 1947 with L. Ahlfors's paper: "Bounded Analytic Functions" (Duke Math. J. 14 (1947) 1-11). Although a number of papers on some of its more elementary properties appeared in the following decade, the concept of analytic capacity remained a mere curiosity until a series of papers by A.G. Vitushkin and M.S. Melnikov employed it to solve several problems in rational approximation theory. For an exposition of that work, see Vitushkin: "Analytic Capacity of Sets and Problems in Approximation Theory" (Russian Mathematical Surveys 22 (1967) 139-200). The question of whether analytic capacity is semiadditive arises naturally in Vitushkin's work, has received a lot of attention since, and is still open. Semiadditivity has been proved in certain special cases by Melnikov: "A Bound for the Cauchy Integral along an Analytic Curve" (Mat. Sb. 71 (113) (1966) 503-515) and by N.A. Shirokov: "Some Properties of Analytic Capacity" (Vestnik Leningrad. Univ. 1 (1972) 77-86). The equivalence of various forms of the semiadditivity problem has been proved by A.M. Davie: "Analytic Capacity and Approximation Problems" (Trans. Amer. Math. Soc. 171 (1972) 409-444).

This thesis is concerned with the more stringent questions of whether analytic capacity is subadditive and whether it is strongly subadditive. We cannot answer either of these questions, but we give positive results in several special cases. Our main tool is the Szegö kernel function, and the approach is constructive and fairly elementary. We use very little functional analysis, except in the "background" Chapters I and II. We make no use of existing work on the semiadditivity problem: indeed the point of view of this thesis is quite different.

Chapters I and II present known material which will be needed later. Chapter I deals with the more elementary properties of analytic capacity, and Chapter II is an exposition of the theory of the Szegö kernel function and related functions.

Chapter III is a study of a perturbation technique. Specifically, we consider the effect, on the kernel functions of a domain, of removing a small disc from that domain. We also take a first look at the connection between the resulting theory and the subadditivity problem. In Chapter IV, we use harmonic functions to study domains whose complements are contained in the real line. As applications of this, we derive explicit formulae for the kernel functions of such domains and deduce some of their properties, we prove an easy subadditivity theorem, and we give a counterexample to a conjecture of Shirokov about the behaviour of the Ahlfors function. Chapter V combines the work of Chapters III and IV to give several theorems on the subadditivity and the strong subadditivity of analytic capacity and on related concepts. Most of the material in Chapters III, IV and V is new.

I am greatly indebted to my present supervisor, Dr. A.M. Davie, for his constant excellent guidance. I should also like to thank Prof. F.F. Bonsall, for his supervision of my work during my first year; Prof. T.W. Gamelin, for some helpful correspondence; and the Carnegie Trust for the Universities of Scotland, for providing financial support throughout my research. Finally I owe a real debt of gratitude to my family and friends, many of whom have played a much more significant part in the production of this thesis than I think they realise.

CONTENTS

CHAPTER I

ELEMENTARY PROPERTIES OF ANALYTIC CAPACITY

The purpose of Chapter I is to give a general introduction to analytic capacity. All the results in it are elementary and well-known. Most of the material in it is needed for later use: though one or two items have been included merely for illustration. Most of the results appear in [10], chapter 8, and in [13], chapter 1.

§1 Notation

The following notation will be used throughout this thesis. $\underset{\sim}{C}$ denotes the complex plane. $\underset{\sim}{R}$ denotes the real line. S^2 denotes the Riemann sphere, i.e. the compactification of $\underset{\sim}{C}$ by the addition of a point ∞. If $z \in \underset{\sim}{C}$ and $r > 0$, then $D(z;r)$ and $\overline{D}(z;r)$ denote respectively the open and closed discs with centre z and radius r. D and $\overline{D}$ denote respectively the open and closed discs with centre 0 and radius 1. If $A \subset \underset{\sim}{C}$ and $z \in \underset{\sim}{C}$, then $d(z,A)$ denotes the distance between z and A; that is:

$$d(z,A) = \inf\{|z-\zeta| : \zeta \in A\}.$$

Let f be a function analytic at ∞. Then by Laurent's theorem f has an expansion of the form $f(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots\ldots$ for large $|z|$. We define:

$$\begin{aligned} f'(\infty) &= a_1 \\ &= \lim_{z\to\infty} z(f(z) - f(\infty)) \\ &= \lim_{z\to\infty} (-z^2 f'(z)) \\ &= \frac{1}{2\pi i}\int_\Gamma f(z)dz \end{aligned}$$

where Γ is any closed curve on and outside which f is analytic. We shall extend this slight abuse of notation even further. If ϕ is any mapping, conformal at ∞, and satisfying $\phi(\infty) = \infty$, then ϕ has an expansion of the form $\phi(z) = a_{-1}z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots\ldots$ for large $|z|$. We define:

$$\begin{aligned}\phi'(\infty) &= a_{-1}\\ &= \lim_{z\to\infty} \frac{\phi(z)}{z}\\ &= \lim_{z\to\infty} \phi'(z).\end{aligned}$$

If V is any subset of S^2, then $\overline{V}$ denotes the closure and ∂V the boundary of V in S^2. Let E be a compact subset of $\underset{\sim}{C}$. $\Omega(E)$ denotes the connected component of $S^2 - E$ containing ∞. The <u>outer boundary</u> of E is $\partial\Omega(E)$. A function f analytic on $\Omega(E)$ is <u>admissible for</u> E if $|f| < 1$ on $\Omega(E)$ and $f(\infty) = 0$.

§2 Analytic Capacity

Let E be a compact subset of $\underset{\sim}{C}$. The <u>analytic capacity</u> of E is:

$$\gamma(E) = \sup\{|f'(\infty)| : f \text{ is admissible for } E\}.$$

Note that we have also:

$$\gamma(E) = \sup\{|f'(\infty)| : f \text{ is analytic on } \Omega(E),\ |f| < 1 \text{ on } \Omega(E)\}.$$

For if f is analytic on $\Omega(E)$ and $|f| < 1$ on $\Omega(E)$, then the function g defined on $\Omega(E)$ by the formula $g(z) = \dfrac{f(z) - f(\infty)}{1 - \overline{f(\infty)}f(z)}$ is admissible for E, and:

$$|g'(\infty)| = \frac{|f'(\infty)|}{1 - |f(\infty)|^2} \geq |f'(\infty)|.$$

2.1 Proposition Let E and F be compact subsets of $\mathbb{C}$, with $E \subset F$. Then $\gamma(E) \leqslant \gamma(F)$.

Proof Clearly $\Omega(F) \subset \Omega(E)$. If f is admissible for E then $f|_{\Omega(F)}$ is admissible for F and so $|f'(\infty)| \leqslant \gamma(F)$. Hence $\gamma(E) \leqslant \gamma(F)$.

If A is any subset of S^2, then the analytic capacity of A is defined as:

$$\gamma(A) = \sup\{\gamma(E) : E \subset A, \ E \text{ is a compact subset of } \mathbb{C}\}.$$

If A is itself a compact subset of $\mathbb{C}$, then this definition is consistent by Proposition 2.1. It now follows from Proposition 2.1 that if $A, B \subset S^2$ and $A \subset B$ then $\gamma(A) \leqslant \gamma(B)$. We shall be concerned almost exclusively with compact subsets of $\mathbb{C}$.

2.2 Proposition Let $E \subset \mathbb{C}$ and $c \in \mathbb{C}$. Then:

(a) $\gamma(E + c) = \gamma(E)$;

(b) $\gamma(cE) = |c|\gamma(E)$;

(c) $\gamma(E^*) = \gamma(E)$

where E^* is the reflection of E in the real axis.

Proof The proofs are easy: we shall prove (c) as an example. We may suppose that E is compact. Let f be admissible for E. For $z \in \Omega(E^*) = \Omega(E)^*$, write $g(z) = \overline{f(\overline{z})}$. Then g is admissible for E^*, and so $\gamma(E^*) \geqslant |g'(\infty)| = |f'(\infty)|$. This holds for all f admissible for E, and so $\gamma(E^*) \geqslant \gamma(E)$. Replacing E by E^* gives $\gamma(E) \geqslant \gamma(E^*)$. Hence $\gamma(E^*) = \gamma(E)$.

Observe that the analytic capacity of a compact subset of $\mathbb{C}$ depends only on its outer boundary: for if E and F are compact and $\partial\Omega(E) = \partial\Omega(F)$ then $\Omega(E) = \Omega(F)$. Thus "holes" inside a set do not alter its analytic capacity.

The next theorem was first proved by S.Y. Havinson ([14], theorem 9). The proof we give is due to S. Fisher [9].

2.3 <u>Theorem</u> Let E be a compact subset of $\underset{\sim}{C}$. Then there is a unique function f admissible for E, the <u>Ahlfors function</u> of E, satisfying $f'(\infty) = \gamma(E)$.

<u>Proof</u> First we show the existence of such an f. Choose a sequence $\{f_n\}$ of functions admissible for E such that $f_n'(\infty) \to \gamma(E)$. $\{f_n\}$ is uniformly bounded (by 1) and is therefore normal. So some subsequence $\{f_{n_r}\}$ converges uniformly on compact subsets of $\Omega(E)$ to some function f. $f(\infty) = \lim_{r\to\infty} f_{n_r}(\infty) = 0$. $|f| \leqslant 1$ on $\Omega(E)$ and so $|f| < 1$ by the open mapping theorem. So f is admissible for E. $f'(\infty) = \lim_{r\to\infty} f_{n_r}'(\infty) = \gamma(E)$ as required.

To show uniqueness of f, suppose that f_0 and f_1 are admissible for E, that $f_0'(\infty) = f_1'(\infty) = \gamma(E)$, and that $f_0 \neq f_1$. Then $f = \frac{1}{2}(f_0 + f_1)$ is admissible for E and $f'(\infty) = \gamma(E)$. Write $h = \frac{1}{2}(f_1 - f_0)$. Then $f_1 = f + h$, $f_0 = f - h$, and $h \neq 0$. $|f|^2 + |h|^2 \pm 2\mathrm{Re}(f\bar{h}) = |f \pm h|^2 < 1$, and so $|f|^2 + |h|^2 < 1$. Write $k = \frac{1}{2}h^2 \neq 0$. $|k| \leqslant \frac{1}{2}(1 - |f|^2) = \frac{1}{2}(1 - |f|)(1 + |f|) < 1 - |f|$; so $|f| + |k| < 1$. For large $|z|$, $k(z) = \frac{a_n}{z^n} + \frac{a_{n+1}}{z^{n+1}} + \ldots\ldots$ for some n, where $a_n \neq 0$. $h(\infty) = 0$ and so $k(\infty) = 0$ and $k'(\infty) = 0$; so $n \geqslant 2$. So, for some $\epsilon > 0$, $\epsilon|a_n||z|^{n-1} \leqslant 1$ in some neighbourhood of E. Write $g = f + \epsilon\bar{a}_n z^{n-1}k$. $|g| \leqslant |f| + |k| < 1$ in some neighbourhood of E; so $|g| < 1$ throughout $\Omega(E)$. Also $g(\infty) = 0$. So g is admissible for E. But $g'(\infty) = f'(\infty) + \epsilon\bar{a}_n a_n > f'(\infty) = \gamma(E)$.

f_E will denote the Ahlfors function of the compact set E.

The following theorem is an easy consequence.

2.4 <u>Theorem</u> Let $\{E_n\}$ be a decreasing sequence of compact subsets of $\underset{\sim}{C}$, and let $E = \bigcap_{n=1}^{\infty} E_n$. Then $f_{E_n} \to f_E$ uniformly on compact subsets of $\Omega(E)$. In particular $\gamma(E_n) \to \gamma(E)$.

<u>Proof</u> $\{f_{E_n}\}$ is a normal sequence. Let h be any cluster point of $\{f_{E_n}\}$. h is admissible for E and so $h'(\infty) \leq \gamma(E)$. But $\gamma(E) \leq \liminf \gamma(E_n) = \liminf f_{E_n}'(\infty) \leq h'(\infty)$. So $h'(\infty) = \gamma(E)$; i.e. $h = f_E$. So f_E is the only cluster point of $\{f_{E_n}\}$. That is, $f_{E_n} \to f_E$.

In the above theorem and proof it is not necessary to assume that $E_{n+1} \subset E_n$. It is sufficient that $\{\Omega(E_n)\}$ should exhaust $\Omega(E)$ from within: that is, $\Omega(E_n) \subset \Omega(E)$ and every compact subset of $\Omega(E)$ is contained in $\Omega(E_n)$ for all sufficiently large n.

2.5 <u>Corollary</u> Let E be a compact subset of $\underset{\sim}{C}$. Then:

$$\gamma(E) = \inf\{\gamma(U) : U \text{ is open, } E \subset U\}.$$

<u>Proof</u> Write $E_n = \{z \in \underset{\sim}{C} : d(z,E) \leq \frac{1}{n}\}$ and $U_n = \{z \in \underset{\sim}{C} : d(z,E) < \frac{1}{n}\}$. E_n is compact, and U_n is open. $\gamma(E) \leq \gamma(U_n) \leq \gamma(E_n) \to \gamma(E)$.

The well-known Schwarz lemma in complex analysis says that if $f : D \to D$ is analytic and $f(0) = 0$ then $f'(0) \leq 1$ and $|f(z)| \leq |z|$ for all $z \in D$. This gives a neat characterisation of the Ahlfors function of a connected set, as follows.

2.6 <u>Theorem</u> Let E be a compact connected subset of $\underset{\sim}{C}$

containing more than one point. Let f be the conformal map of $\Omega(E)$ onto D satisfying $f(\infty) = 0$ and $f'(\infty) > 0$. Then $\gamma(E) = f'(\infty)$ and f is the Ahlfors function of E.

Proof f is admissible for E and so $f'(\infty) \leq \gamma(E)$. Let h be admissible for E. $h \circ f^{-1}$ is analytic on D, is bounded by 1, and vanishes at 0. By Schwarz's lemma, $|(h \circ f^{-1})'(0)| \leq 1$; that is, $|h'(\infty)| \leq f'(\infty)$. Since this holds for all h admissible for E, $\gamma(E) \leq f'(\infty)$. So $\gamma(E) = f'(\infty)$. By Theorem 2.3, f is the Ahlfors function of E.

Theorem 2.6 is really just a re-wording of Schwarz's lemma. Indeed, the whole theory of analytic capacity can be considered as a generalisation of Schwarz's lemma to multiply-connected domains. It was this view which motivated the initial research by L. Ahlfors [1].

2.7 Corollary

(a) The analytic capacity of a disc is its radius.

(b) The analytic capacity of a straight-line segment is a quarter of its length.

Proof (a) Write $f(z) = \frac{1}{z}$. f maps $\Omega(\bar{D})$ conformally onto D, $f(\infty) = 0$, and $f'(\infty) = 1$. So, by Theorem 2.6, $\gamma(\bar{D}) = 1$. The result follows for all closed discs by Proposition 2.2, and for open discs by monotonicity of γ.

(b) Write $g(z) = z + \frac{1}{z}$ for $z \in D$. g maps D conformally onto $\Omega[-2,2]$ and $g(0) = \infty$. Its inverse f maps $\Omega[-2,2]$ conformally onto D; $f(\infty) = 0$, and $f'(\infty) = 1$. So $\Omega[-2,2] = 1$. Use Proposition 2.2 and monotonicity.

2.8 Corollary Let E be a compact subset of $\mathbb{C}$, and let

$\gamma(E) = 0$. Then E is totally disconnected.

Proof The analytic capacity of a continuum is positive by Theorem 2.6.

The next proposition shows that the Ahlfors function of a set behaves as one might expect under a conformal mapping of the complement.

2.9 Proposition Let E_1 and E_2 be compact subsets of $\underset{\sim}{C}$ with Ahlfors functions f_1 and f_2 respectively. Suppose there is a conformal map ϕ of $\Omega(E_2)$ onto $\Omega(E_1)$ with $\phi(\infty) = \infty$ and $\phi'(\infty) = 1$. Then $\gamma(E_2) = \gamma(E_1)$ and $f_2 = f_1 \circ \phi$.

Proof $f_1 \circ \phi$ is admissible for E_2 and $(f_1 \circ \phi)'(\infty) = f_1'(\infty) = \gamma(E_1)$. So $\gamma(E_1) \leqslant \gamma(E_2)$. Similarly, by considering ϕ^{-1}, we find that $\gamma(E_2) \leqslant \gamma(E_1)$; so $\gamma(E_2) = \gamma(E_1)$. Since $f_1 \circ \phi$ is admissible for E_2 and $(f_1 \circ \phi)'(\infty) = \gamma(E_2)$, $f_1 \circ \phi = f_2$.

The next theorem gives two characterisations of compact sets with zero analytic capacity.

2.10 Theorem Let E be a compact subset of $\underset{\sim}{C}$. Then the following are equivalent:

(a) $\gamma(E) = 0$;

(b) every bounded analytic function on $S^2 - E$ is constant;

(c) E is a removable singularity for bounded analytic functions: that is, whenever U is a domain in S^2 and f is a bounded analytic function on $U - E$, then f extends to a bounded analytic function on U.

Proof (a) $\Rightarrow$ (b) Let f be a bounded analytic function on $S^2 - E$ with $f(\infty) = 0$. Let $z_0 \in S^2 - E$. Write $g(z) = \dfrac{f(z) - f(z_0)}{z - z_0}$. g is analytic and bounded on $S^2 - E$ and $g(\infty) = 0$; so some positive multiple of g is admissible for E. Since $\gamma(E) = 0$, $g'(\infty) = 0$; that is, $f(z_0) = 0$. z_0 was arbitrary and so $f = 0$ throughout $S^2 - E$.

(b) $\Rightarrow$ (a) Extend f_E to $S^2 - E$ by defining it to be 0 on the bounded components of $S^2 - E$, if any exist. f_E is constant and so $\gamma(E) = f_E'(\infty) = 0$.

(c) $\Rightarrow$ (b) Let f be a bounded analytic function on $S^2 - E$. By assumption, f extends to a bounded analytic function on S^2. By Liouville's theorem f is constant.

(a) & (b) $\Rightarrow$ (c) Let U be a domain in S^2 and let f be a bounded analytic function on $U - E$. Let $z_0 \in E$. $\gamma(E) = 0$ and so E is totally disconnected by Corollary 2.8. Hence we can find Jordan curves Γ_1 and Γ_2, of a type to which Cauchy's theorem applies, so that Γ_2 encloses z_0, Γ_1 encloses Γ_2, and Γ_1, Γ_2 and the region between them lie in $U - E$. Write:

$$f_1(\zeta) = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{f(z)dz}{z - \zeta} \qquad (\zeta \text{ inside } \Gamma_1)$$

$$f_2(\zeta) = -\frac{1}{2\pi i}\int_{\Gamma_2} \frac{f(z)dz}{z - \zeta} \qquad (\zeta \text{ outside } \Gamma_2).$$

By Cauchy's theorem, $f = f_1 + f_2$ between Γ_1 and Γ_2. Let V be the inside of Γ_1. Continue f_2 on $V - E$ by the formula $f_2 = f - f_1$; then f_2 is analytic on $S^2 - E$. Now choose any Jordan curve Γ inside Γ_1 and enclosing Γ_2. On and outside Γ, f_2 is bounded. Inside Γ, f_1 is bounded and so $f_2 = f - f_1$ is bounded wherever it is defined. Thus f_2 is a bounded analytic function on $S^2 - E$. By assumption, f_2 is constant: $f_2 = f_2(\infty) = 0$. Hence $f = f_1$ on $V - E$

and so f extends to the analytic function f_1 on V. Thus f extends to be analytic on some neighbourhood of every point of E, and therefore extends to be analytic on U.

Theorem 2.10 applies equally well to compact subsets of S^2. The extension is easy.

2.11 Corollary Let E and F be compact subsets of $\underset{\sim}{C}$ and let $\gamma(E) = 0$. Then $\gamma(E \cup F) = \gamma(F)$.

Proof The Ahlfors function of $E \cup F$ extends to be analytic on $\Omega(F)$ by Theorem 2.10.

§3 Estimates of Analytic Capacity

Let E be a compact subset of $\underset{\sim}{C}$. A regular neighbourhood of E is a bounded open set, containing E, whose boundary is the union of finitely many pairwise disjoint analytic Jordan curves. The Painlevé length of E, denoted by $\kappa(E)$, is the infimum of the set of real numbers ℓ with the property that every open set containing E contains a regular neighbourhood of E whose boundary has length at most ℓ. If that set is empty then we write $\kappa(E) = \infty$. Our first proposition gives an estimate for $\gamma(E)$ in terms of $\kappa(E)$.

3.1 Proposition Let E be a compact subset of $\underset{\sim}{C}$. Then $\gamma(E) \leq \dfrac{\kappa(E)}{2\pi}$.

Proof We may assume that $\kappa(E) < \infty$. Let $\epsilon > 0$. Surround E by a union Γ of finitely many analytic Jordan curves of total length at most $\kappa(E) + \epsilon$. Then $\gamma(E) = f_E'(\infty) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma f_E(z)\,dz \leq \dfrac{1}{2\pi}\int_\Gamma |f_E(z)|\,ds$ $\leq \dfrac{\kappa(E) + \epsilon}{2\pi}$. This holds for all $\epsilon > 0$ and so $\gamma(E) \leq \dfrac{\kappa(E)}{2\pi}$.

If E is a bounded subset of $\underset{\sim}{C}$, then E is contained in a disc of radius $\operatorname{diam} E$, so that $\gamma(E) \leq \operatorname{diam} E$. However, we can improve on this.

3.2 <u>Proposition</u> Let E be any subset of $\underset{\sim}{C}$. Then $\gamma(E) \leq \frac{1}{2}\operatorname{diam} E$.

<u>Proof</u> We may assume that E is compact. Since the diameters of a set and its convex hull are equal, we may assume that E is convex. It is clear that the Painlevé length of a compact convex set is just its circumference. $\gamma(E) \leq \dfrac{\kappa(E)}{2\pi} = \dfrac{\operatorname{circ}(E)}{2\pi} \leq \frac{1}{2}\operatorname{diam} E$ by the isoperimetric inequality for convex sets ([7], p. 89).

There is also a lower estimate for $\gamma(E)$ in terms of area, which should be mentioned. If E is compact and has positive area, then $f(\zeta) = \displaystyle\iint_E \frac{dxdy}{\zeta - z}$ is a non-constant bounded analytic function on $S^2 - E$ and so $\gamma(E) > 0$. In fact it may be shown ([2], pp. 106 - 107, or [13] pp. 79 - 80 in modern notation) that $f'(\infty) \geq \left(\dfrac{\operatorname{Area} E}{\pi}\right)^{\frac{1}{2}} \|f\|_\infty$, so that $\gamma(E) \geq \left(\dfrac{\operatorname{Area} E}{\pi}\right)^{\frac{1}{2}}$.

The above estimates $\gamma(E) \leq \dfrac{\kappa(E)}{2\pi}$, $\gamma(E) \leq \dfrac{\operatorname{diam} E}{2}$, and $\gamma(E) \geq \left(\dfrac{\operatorname{Area} E}{\pi}\right)^{\frac{1}{2}}$ are all sharp, as equality holds in all three cases if E is a disc.

The next proposition gives estimates for admissible functions.

3.3 <u>Proposition</u> Let E be a compact subset of $\underset{\sim}{C}$ with analytic capacity γ. Let $\zeta \in \underset{\sim}{C} - E$ and let $d(\zeta, E) = r$. Then:

(a) $|g(\zeta)| \leq \dfrac{\gamma}{r}$ whenever g is analytic and bounded by 1 on $\underset{\sim}{C} - E$ and $g(\infty) = 0$;

(b) $|g'(\zeta)| \leq \dfrac{\gamma}{r^2}$ whenever g is analytic and bounded by 1 on $\underset{\sim}{C} - E$.

Proof Let g be analytic and bounded by 1 on $\underset{\sim}{C}-E$. Define $h(z) = \left(\frac{r}{z-\zeta}\right)\left(\frac{g(z)-g(\zeta)}{1-\overline{g(\zeta)}g(z)}\right)$. h is analytic on $\underset{\sim}{C}-E$: the zero in the denominator at $z=\zeta$ is cancelled by a zero in the numerator. The right-hand factor is bounded in modulus by 1, and so $|h(z)| < 1$ whenever $|z-\zeta| \geqslant r$. If $a < r$, then $|h(z)| \leqslant \frac{r}{a}$ whenever $|z-\zeta| = a$; so by the maximum modulus theorem $|h(z)| \leqslant 1$ whenever $|z-\zeta| < r$. So $|h| < 1$ on $\underset{\sim}{C}-E$. $h(\infty) = 0$ and so h is admissible for E.

If now $g(\infty) = 0$, then $r|g(\zeta)| = |h'(\infty)| \leqslant \gamma$. Hence (a). Also, for any g, $r|g'(\zeta)| \leqslant \frac{r|g'(\zeta)|}{1-|g(\zeta)|^2} = |h(\zeta)| \leqslant \frac{\gamma}{r}$ by (a). Hence (b).

The following examples show that, if $\gamma < r$, then the above bounds are best-possible. We shall take $r = 1$; the result for general r follows by a homogeneity argument. $\gamma < 1$ and so we can find a compact subset E of the unit circle with analytic capacity γ. Write $g(z) = \overline{f_E\left(\frac{1}{\bar{z}}\right)}$. g is analytic off E, $g(0) = 0$, and $g'(0) = f_E{}'(\infty) = \gamma$; so g attains the bound (b). $\frac{g(z)}{z}$ is analytic off E, is bounded by 1 (as in the proof of Proposition 3.3), vanishes at ∞, and takes, at 0, the value $g'(0) = \gamma$, attaining the bound (a).

If $\gamma \geqslant r$, then the trivial bounds $|g(z)| \leqslant 1$ and $|g'(z)| \leqslant \frac{1}{r}$ are best-possible. To see this, let E be any compact subset of $\underset{\sim}{C}$, not meeting $D(0;r)$, containing the circle $\{\zeta : |\zeta| = r\}$, and having analytic capacity γ. Then the function equal to 1 on $D(0;r)$ and 0 on the rest of $\underset{\sim}{C}-E$ satisfies the first bound. The function equal to $\frac{z}{r}$ on $D(0;r)$ and 0 on the rest of $\underset{\sim}{C}-E$ satisfies the second bound.

3.4 Corollary Let $0 < r < R$. Let E be a subset of the annulus $\{z : r \leq |z| \leq R\}$. Denote by E_* the inversion of E in the unit circle. Then $r^2\gamma(E_*) \leq \gamma(E) \leq R^2\gamma(E_*)$.

<u>Proof</u> We may assume that E is compact. The function $\overline{f_{E_*}\left(\frac{1}{\bar z}\right)}$ is analytic and bounded by 1 on $\Omega(E)$, and so its derivative at 0, $\gamma(E_*)$, is at most $\frac{\gamma(E)}{r^2}$ by Proposition 3.3 (b). The other bound is proved similarly.

§4 <u>Logarithmic Capacity</u>

In this section, which is really a digression, we look at another set function, the logarithmic capacity, which historically precedes analytic capacity. We shall take the liberty of giving two definitions of it and not proving their equivalence, since the matter is treated in detail in several textbooks (see, for example, [24] chapter 3). The importance of logarithmic capacity for our purpose is its equivalence to analytic capacity in the case of compact connected sets (Proposition 4.1).

Let μ be a probability measure (i.e. a positive Borel measure with total mass 1) on $\underset{\sim}{C}$. We define:

$$I(\mu) = \iint \log\frac{1}{|z-\zeta|}d\mu(z)d\mu(\zeta).$$

Let E be a compact subset of $\underset{\sim}{C}$. We define:

$$V(E) = \inf\{I(\mu) : \mu \text{ is a probability measure supported on } E\}.$$

Then $-\infty < V(E) \leq \infty$. The <u>logarithmic capacity</u> of E is defined as:

$$\operatorname{cap}(E) = \begin{cases} e^{-V(E)} & \text{if } V(E) < \infty \\ 0 & \text{if } V(E) = \infty. \end{cases}$$

It is easily verified that the analogues of Propositions 2.1 and 2.2

for logarithmic capacity are true.

We can also define $\operatorname{cap}(E)$ as the supremum of $|f'(\infty)|$ over all functions analytic, but not necessarily single-valued, on $\Omega(E)$ which satisfy:

(a) $|f|$ is single-valued;

(b) $|f| < 1$ on $\Omega(E)$;

(c) $f(\infty) = 0$.

For the equivalence of these two definitions see [25] p. 134. For other equivalent definitions see [24] chapter 3.

4.1 Proposition Let E be a compact subset of $\mathbb{C}$. Then $\operatorname{cap}(E) \geqslant \gamma(E)$, with equality if E is connected.

Proof It is obvious from the second definition that $\operatorname{cap}(E) \geqslant \gamma(E)$. If E is connected then $\Omega(E)$ is simply-connected, so that any multiple-valued function analytic on $\Omega(E)$ separates into single-valued ones. Then the second definition of $\operatorname{cap}(E)$ above reduces to the definition of $\gamma(E)$.

4.2 Proposition Let E be a compact subset of $\mathbb{C}$. Let q be a function from E into $\mathbb{C}$ such that $|q(z) - q(\zeta)| \leqslant |z - \zeta|$ for all $z, \zeta \in E$. Then $\operatorname{cap}(q(E)) \leqslant \operatorname{cap}(E)$.

Proof q is continuous and so $q(E)$ is compact. Let μ be a probability measure on E. Then $q^{-1}(\mu)$ is a probability measure on $q(E)$, and so $V(q(E)) \geqslant I(q^{-1}(\mu)) = \iint \log \frac{1}{|q(z)-q(\zeta)|} d\mu(z)d\mu(\zeta)$ $\geqslant \iint \log \frac{1}{|z-\zeta|} d\mu(z)d\mu(\zeta) = I(\mu)$. This holds for all μ, so that $V(q(E)) \geqslant V(E)$. So $\operatorname{cap}(q(E)) \leqslant \operatorname{cap}(E)$.

It is worth noting that the analogue of Proposition 4.2

for analytic capacity is false. We shall see in Chapter IV, for example, that the analytic capacity of the union of a closed interval $I \subset \mathbb{R}$ and a small disc centred at $x + iy$ $(x \notin I,\ y \neq 0)$ is smaller than the analytic capacity of the union of I and a disc of the same radius centred at x. (See Corollary 16.3.)

4.3 Corollary

(a) Let E be a compact connected subset of $\mathbb{C}$. Then $\gamma(E) = \operatorname{cap}(E) \geq \frac{1}{4} \operatorname{diam} E$.

(b) Let Γ be a rectifiable arc in $\mathbb{C}$. Then $\gamma(\Gamma) = \operatorname{cap}(\Gamma) \leq \frac{1}{4} \ell(\Gamma)$.

Proof (a) Let q be the projection onto a line parallel to a diameter of E. Then $\operatorname{cap}(q(E)) \leq \operatorname{cap}(E)$ by Proposition 4.2. But $q(E)$ is a line segment of length $\operatorname{diam} E$, and so $\operatorname{cap}(q(E)) = \gamma(q(E)) = \frac{1}{4} \operatorname{diam} E$ by Corollary 2.7 (b).

(b) Let $q : [0, \ell(\Gamma)] \to \mathbb{C}$ be a parametrisation of Γ by length. Then $|q(x) - q(y)| \leq |x - y|$ $(x, y \in [0, \ell(\Gamma)])$. So by Proposition 4.2, $\gamma(\Gamma) = \operatorname{cap}(\Gamma) \leq \operatorname{cap}[0, \ell(\Gamma)] = \frac{1}{4}\ell(\Gamma)$.

The fact that $\gamma(E) \geq \frac{1}{4} \operatorname{diam} E$ if E is compact and connected is really just a restatement of the $\frac{1}{4}$-theorem of Koebe: see [10], p. 199. Corollary 4.3 implies that, among all arcs with a given capacity, a straight-line segment has the greatest diameter and the smallest length. Corollary 4.3 (b) was first shown by M. Fekete [8]. The result has been generalised to all compact connected sets by Ch. Pommerenke [19].

§5 The T_ϕ Operator

The T_ϕ operator is a basic tool introduced by Vitushkin for the purpose of studying rational approximation. We shall need it on only one occasion, but its use is such an important technique that we shall derive its properties rather than merely quote them. Throughout this section, "function" will mean "complex-valued function".

Let V be an open subset of $\underset{\sim}{C}$. Let g be a continuously differentiable function on V. We define:

$$\frac{\partial g}{\partial z} = \frac{1}{2}\left(\frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y}\right);$$

$$\frac{\partial g}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y}\right).$$

It is easy to verify that the rules for differentiating products and sums with respect to z or $\bar{z}$ are formally the same as the corresponding rules for functions of one real variable: e.g. $\frac{\partial}{\partial z}(fg) = f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}$. If g is analytic, then the Cauchy-Riemann equations show that $\frac{\partial g}{\partial \bar{z}} = 0$ and that $\frac{\partial g}{\partial z} = g'$.

Let V be a domain in $\underset{\sim}{C}$ bounded by a piecewise-continuously-differentiable contour in $\underset{\sim}{C}$. Let g be a continuously differentiable function on a neighbourhood of $\bar{V}$. Green's theorem states that:

$$\iint_V \frac{\partial g}{\partial x}\,dxdy = \int_{\partial V} g\,dy;$$

$$\iint_V \frac{\partial g}{\partial y}\,dxdy = -\int_{\partial V} g\,dx.$$

Multiplying these by $\frac{1}{2}$ and $\frac{1}{2}i$ respectively, and adding, we have:

$$\iint_V \frac{\partial g}{\partial \bar{z}}\,dxdy = \frac{1}{2i}\int_{\partial V} g\,dz.$$

This is the complex form of Green's theorem. It reduces to Cauchy's theorem if g is analytic.

Now let U be a domain in $\underset{\sim}{C}$ bounded by a piecewise-continuously-differentiable contour Γ. Let f be a continuously differentiable function on a neighbourhood of $\overline{U}$, and let $\zeta \in U$. Applying Green's theorem to the function $\dfrac{f(z)}{z-\zeta}$ on the domain $U - \overline{D}(\zeta;\epsilon)$, where $\epsilon < d(\zeta, \partial U)$, gives:

$$\frac{1}{\pi}\iint\limits_{U-\overline{D}(\zeta;\epsilon)} \frac{\partial f}{\partial \bar z}\,\frac{1}{z-\zeta}\,dxdy = \frac{1}{2\pi i}\int\limits_{\Gamma}\frac{f(z)}{z-\zeta}\,dz - \frac{1}{2\pi i}\int\limits_{|z-\zeta|=\epsilon}\frac{f(z)}{z-\zeta}\,dz.$$

Letting $\epsilon \to 0$ gives, on rearranging the terms:

$$f(\zeta) = \frac{1}{2\pi i}\int\limits_{\Gamma}\frac{f(z)}{z-\zeta}\,dz - \frac{1}{\pi}\iint\limits_{U}\frac{\partial f}{\partial \bar z}\,\frac{1}{z-\zeta}\,dxdy.$$

This is Green's formula for f. It is easy to verify, using polar co-ordinates, that $\dfrac{1}{z-\zeta}$ is locally integrable, so that the last integral makes sense. Green's formula reduces to Cauchy's formula if f is analytic.

5.1 <u>Lemma</u> Let E be a measurable subset of $\underset{\sim}{C}$. Then for all $\zeta \in \underset{\sim}{C}$:

$$\int\limits_{E}\frac{dxdy}{|z-\zeta|} \leqslant 2\,(\pi \text{ Area } E)^{\frac{1}{2}}.$$

<u>Proof</u> We may assume that $0 < \text{Area } E < \infty$, as otherwise there is nothing to prove. Let $R = (\text{Area } E/\pi)^{\frac{1}{2}}$. Let $\Delta = \overline{D}(\zeta;R)$. Area E = Area Δ and so $\text{Area}(E-\Delta) = \text{Area}(\Delta - E)$. Since $\dfrac{1}{|z-\zeta|} \leqslant \dfrac{1}{R}$ on $E-\Delta$ and $\dfrac{1}{|z-\zeta|} \geqslant \dfrac{1}{R}$ on $\Delta - E$, it follows that:

$$\int\limits_{E-\Delta}\frac{dxdy}{|z-\zeta|} \leqslant \int\limits_{\Delta-E}\frac{dxdy}{|z-\zeta|}\,.$$

Adding $\displaystyle\int_{E\cap\Delta} \frac{dxdy}{|z-\zeta|}$ to both sides gives:

$$\begin{aligned}\int_E \frac{dxdy}{|z-\zeta|} &\leq \int_\Delta \frac{dxdy}{|z-\zeta|}\\ &= \int_0^{2\pi} d\theta \int_0^R \frac{1}{r} r\, dr\\ &= 2\pi R\\ &= 2(\pi \text{ Area } E)^{\frac{1}{2}}.\end{aligned}$$

The following lemma gives the definition and the main properties of the T_ϕ operator. The proof we give is the one in [11], pp. 4-5.

5.2 Lemma Let ϕ be a continuously differentiable function on $\mathbb{C}$, with compact support X. For every bounded measurable function f on $\mathbb{C}$, define a function $T_\phi f$ on $\mathbb{C}$ by:

$$(T_\phi f)(\zeta) = \frac{1}{\pi}\iint \frac{f(z) - f(\zeta)}{z - \zeta} \frac{\partial\phi}{\partial\bar{z}}\, dxdy\,.$$

Then:

(a) $\displaystyle (T_\phi f)(\zeta) = \phi(\zeta)f(\zeta) + \frac{1}{\pi}\iint \frac{f(z)}{z-\zeta}\frac{\partial\phi}{\partial\bar{z}}\, dxdy\,;$

(b) $T_\phi f$ is a bounded measurable function, and

$$\begin{aligned}\|T_\phi f\|_\infty &\leq 2(\text{Area } X / \pi)^{\frac{1}{2}} \left\|\frac{\partial\phi}{\partial\bar{z}}\right\|_\infty \sup_{z,\zeta\in X} |f(z) - f(\zeta)|\\ &\leq 4(\text{Area } X / \pi)^{\frac{1}{2}} \left\|\frac{\partial\phi}{\partial\bar{z}}\right\|_\infty \|f\|_X\,;\end{aligned}$$

(c) $T_\phi f$ is analytic off X and continuous off int X, and vanishes at ∞;

(d) $T_\phi f$ is continuous wherever f is continuous;

(e) $T_\phi f$ is analytic wherever f is analytic;

(f) $f - T_\phi f$ is analytic on the interior of $\phi^{-1}\{1\}$.

Proof (a) follows from the definition of $T_\phi f$ by Green's formula, since ϕ vanishes on any sufficiently large circle centred at 0.

The integral $\displaystyle\iint \frac{f(z)}{z-\zeta}\frac{\partial\phi}{\partial\bar{z}}\,dxdy$, as a function of ζ, is the convolution of the locally integrable function $1/z$ and a bounded function with compact support, and so is continuous. (a) therefore shows that $T_\phi f$ is measurable and is continuous wherever f is continuous.

(c) and (f) follow immediately from (a).

Let $\zeta \in X$. From the definition of $T_\phi f$:

$$|(T_\phi f)(\zeta)| \leq \frac{1}{\pi}\left\|\frac{\partial\phi}{\partial\bar{z}}\right\|_\infty \sup_{z,\zeta\in X}|f(z)-f(\zeta)| \iint_X \frac{dxdy}{|z-\zeta|}$$

$$\leq 2(\text{Area } X/\pi)^{\frac{1}{2}}\left\|\frac{\partial\phi}{\partial\bar{z}}\right\|_\infty \sup_{z,\zeta\in X}|f(z)-f(\zeta)|$$

by Proposition 5.1. The same estimate holds for all $\zeta \in \underset{\sim}{C}$ by (c). That proves (b).

Finally, suppose that f is analytic on a disc Δ. Since $\dfrac{f(z)-f(\zeta)}{z-\zeta}$, as a function of the two variables z and ζ, is analytic on $\Delta\times\Delta$, the integral $\displaystyle\iint_\Delta \frac{f(z)-f(\zeta)}{z-\zeta}\frac{\partial\phi}{\partial\bar{z}}\,dxdy$ is analytic on Δ. But $\displaystyle\iint_{\underset{\sim}{C}-\Delta} \frac{f(z)-f(\zeta)}{z-\zeta}\frac{\partial\phi}{\partial\bar{z}}\,dxdy$ is also analytic on Δ, and so $T_\phi f$ is analytic on Δ. That proves (e).

Lemma 5.2 can be used to prove results in rational approximation theory. The following corollary illustrates the technique.

5.3 Corollary Let $z_0 \in \underset{\sim}{C}$, and let f be a bounded measurable function on $\underset{\sim}{C}$ which is continuous at z_0. Then f is the uniform limit of a sequence $\{f_n\}$ of bounded measurable functions each of which is continuous wherever f is continuous, analytic wherever f is analytic, and analytic at z_0.

<u>Proof</u> We may assume that $z_0 = 0$. Let h be any continuously differentiable function on $[0,\infty[$ satisfying $h'(0) = 0$. Write $\phi(re^{i\theta}) = h(r)$. Then ϕ is continuously differentiable on $\underset{\sim}{C}$ and $\frac{\partial\phi}{\partial\bar{z}} = \frac{1}{2}h'(r)e^{i\theta}$. Using that fact it is easy to construct, for each positive integer n, a circularly symmetric continuously differentiable function ϕ_n on $\underset{\sim}{C}$ such that $\phi_n(z) = 0$ when $|z| \geqslant \frac{2}{n}$, $\phi_n(z) = 1$ when $|z| \leqslant \frac{1}{n}$, and $\left\|\frac{\partial\phi_n}{\partial\bar{z}}\right\|_\infty \leqslant n$. Lemma 5.2 shows that $T_{\phi_n}f$ is a bounded measurable function which is continuous wherever f is continuous, and analytic wherever f is analytic, that $f - T_{\phi_n}f$ is analytic on $D(0;\frac{1}{n})$, and that:

$$\|T_{\phi_n}f\|_\infty \leqslant 4 \sup_{|z|,|\zeta|\leqslant\frac{1}{n}} |f(z) - f(\zeta)|$$

$$\to 0 \quad \text{as} \quad n \to \infty.$$

Write $f_n = f - T_{\phi_n}f$.

In the proof of Corollary 5.3 the T_ϕ operator is used to express f as the sum of two functions, $f - T_{\phi_n}f$ and $T_{\phi_n}f$, the first of which is analytic on $\{z : |z| < \frac{1}{n}\}$ and the second of which is analytic on $\{z : |z| > \frac{2}{n}\}$. Compare the use of the Cauchy integral in the proof of Theorem 2.10. The integral $\frac{1}{\pi}\iint \frac{f(z)}{z-\zeta}\frac{\partial\phi}{\partial\bar{z}}\,dxdy$ implicit in the proof of Corollary 5.3 can be thought of as a diffuse version of the Cauchy integral, the integration being spread out over the annulus $A = \{z : \frac{1}{n} < |z| < \frac{2}{n}\}$ instead of taking place round a single circle centred at 0. If f is analytic on A then the splittings of f given by the Cauchy integral and the T_ϕ operator are identical.

CHAPTER II

RELATED EXTREMAL PROBLEMS

This chapter examines other extremal problems which are related to the one we studied in Chapter I. Again, nothing in this chapter is new. The material in §7 and §8 was first studied by P. Garabedian [12], and accounts of it are given by S. Bergman in [3], chapter 7, and by J. Garnett in [13], pp. 18 - 23. The approach we take is a combination of the approaches of the two last-mentioned sources: for the sake of applications in later chapters we do not initially single out ∞ as a special point in the way that Garnett does, but we use his modern functional-analysis methods in preference to Bergman's older classical treatment. The extension to arbitrary domains in §9 is recent and is due to N. Suita [23].

§6 Hardy Spaces

The main tool in the study of extremal problems on a domain Ω is the theory of the Hardy spaces $H^p(\Omega)$. As the definitions and properties of the Hardy spaces on the unit disc D are well known, we shall assume a knowledge of that theory. Excellent books on the subject are [6] and [16]. We shall, however, define $H^p(\Omega)$ for more general domains Ω and derive those properties we shall need.

G will denote the class of all domains in S^2 whose boundary is the union of n pairwise disjoint analytic Jordan curves in $\underset{\sim}{C}$ for some positive integer n.

An unfortunate anomaly has arisen, for historical reasons, in the question of the direction of contour integration. Suppose $\Omega \in G$. If $\infty \notin \Omega$ and Ω is simply connected, then $\partial\Omega$ is always traversed anti-clockwise. If $\infty \notin \Omega$ but Ω is multiply-connected, then the outer boundary of $\bar{\Omega}$ is traversed anti-clockwise, so that the remaining components of the boundary must be taken clockwise. Thus Ω is always to the left of $\partial\Omega$. Logically, then, if $\infty \in \Omega$ then all the components of $\partial\Omega$ should be taken clockwise: but convention decrees otherwise. Reluctantly we shall bow to convention, run anti-clockwise, and put up with the alternation in sign which will occur so often when we change from domains containing ∞ to domains not containing ∞.

Throughout this section $\Omega \in G$. Denote by $A(\Omega)$ the uniform algebra of all complex-valued functions continuous on $\bar{\Omega}$ which are analytic in Ω. We shall make the obvious identification between a function in $A(\Omega)$ and its restriction to $\partial\Omega$, and we shall denote the algebra of such restrictions by $A(\Omega)$ also.

Now let $1 \leqslant p \leqslant \infty$. L^p will denote $L^p(\mu)$, where μ is arc length on $\partial\Omega$. We define $H^p(\Omega)$ as the closure of $A(\Omega)$ in L^p (the weak-* closure in the case $p = \infty$). We shall abbreviate $H^p(\Omega)$ to H^p and $A(\Omega)$ to A when it is clear which domain is meant. If $\Omega^{(1)}, \Omega^{(2)} \in G$ and if there is a conformal mapping ϕ from $\Omega^{(2)}$ onto $\Omega^{(1)}$, then ϕ continues analytically across $\partial\Omega^{(2)}$, and it is easy to check that $h \circ \phi \in H^p(\Omega^{(2)})$ for each $h \in H^p(\Omega^{(1)})$. In other words, H^p is conformally invariant to the extent to which we have defined it. Clearly $H^p \subset H^r$ whenever $1 \leqslant r \leqslant p \leqslant \infty$.

Suppose $f \in H^p$, so that $f_n \to f$ in L^p for some sequence $\{f_n\}$ in A. By Cauchy's formula, for all $\zeta \in \Omega$:

$$f_n(\zeta) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\partial\Omega} \dfrac{f_n(z)dz}{z - \zeta} & \text{if } \infty \notin \Omega \\ \dfrac{1}{2\pi i}\displaystyle\int_{\partial\Omega} f_n(z)\left(\dfrac{1}{z-a} - \dfrac{1}{z-\zeta}\right)dz & \text{if } \infty \in \Omega \end{cases}$$

where a is any point of $S^2 - \bar{\Omega}$. The right hand side converges: denote its limit by $\tilde{f}(\zeta)$. Then:

$$(1) \qquad \tilde{f}(\zeta) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\partial\Omega} \frac{f(z)dz}{z-\zeta} & \text{if } \infty \notin \Omega \\ \tilde{f}(\infty) - \dfrac{1}{2\pi i}\displaystyle\int_{\partial\Omega} \frac{f(z)dz}{z-\zeta} & \text{if } \infty \in \Omega. \end{cases}$$

Fatou's theorem (which follows very easily from the statement of it for the unit disc) says that f is the almost-everywhere non-tangential limit of $\tilde{f}$.

We shall need the following simple properties of H^p.

6.1 Theorem If $f \in H^\infty$ then $\tilde{f}$ is a bounded analytic function on Ω and $\sup|\tilde{f}| = \|f\|$. Moreover, every bounded analytic function on Ω can be obtained in that way.

Proof Define a linear mapping Ψ from H^∞, with its weak-* topology, into the space of analytic functions on Ω, with the topology of uniform convergence on compact sets, by $\Psi(f) = \tilde{f}$. Ψ is one-one by Fatou's theorem. It is therefore sufficient to show that Ψ maps the closed unit ball B of H^∞ onto the set C of analytic functions on Ω bounded by 1. If $\{\tilde{f}_\alpha\}$ is any net in B converging (weak-*) to f, then the Cauchy integral (1) shows that $\{f_\alpha\}$ is uniformly bounded on compact subsets of Ω, and converges pointwise (and hence uniformly on compact sets) to the function $\tilde{f}$. This shows that Ψ is continuous on B. Denote the closed unit ball of A by A_1. Ψ maps A_1, which is a dense subset of B, onto A_1, which is a dense subset of the closed set C. Since B is compact by the Banach-Alaoglu theorem ([5], theorem V.4.2), $\Psi(B) = C$.

In future, if $f \in H^p$, we shall identify f and $\tilde{f}$. The last theorem then identifies H^∞ with the space of bounded analytic functions on Ω (and its proof shows, incidentally, that the weak-* topology and the topology of uniform convergence on compact sets coincide on the unit ball of H^∞). The next theorem gives an analogous characterisation of H^p when $1 \leq p < \infty$.

6.2 <u>Theorem</u> Let $1 \leq p < \infty$. Let f be analytic on Ω. Then the following are equivalent:

(a) $f \in H^p$;

(b) there is a real-valued harmonic function u on Ω with $|f|^p \leq u$;

(c) there is an open set V containing $\partial\Omega$ and a real-valued harmonic function u on $\Omega \cap V$ with $|f|^p \leq u$ on $\Omega \cap V$.

<u>Proof</u> The result is well-known in the case when Ω is the unit disc, and it follows for all simply-connected $\Omega \in G$ by the conformal invariance of (a), (b) and (c). In the general case, denote the components of $\partial\Omega$ by $\Gamma_1,\dots,\Gamma_n$ and write $\Omega = \bigcap_{j=1}^{n} \Omega_j$, where $\partial\Omega_j = \Gamma_j$. Using the Cauchy integral (as in the proof of Theorem 2.10) we can write $f = \sum_{j=1}^{n} f_j$, where f_j is analytic on Ω_j, and it is clear that $f \in H^p(\Omega)$ if and only if $f_j \in H^p(\Omega_j)$ for $j = 1,\dots,n$.

(a) $\Rightarrow$ (b) $f_j \in H^p(\Omega_j)$ and so by the simply-connected case there is a harmonic function u_j on Ω_j such that $|f_j|^p \leq u_j$ on Ω_j. So $|f|^p \leq n^{p-1}\sum_{j=1}^{n}|f_j|^p \leq n^{p-1}\sum u_j$.

(b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (a) We may assume that $V = \cup V_j$ where the V_j are disjoint open sets and $\Gamma_j \subset V_j$ $(j = 1,\dots,n)$. On $\Omega \cap V_j$, $|f|^p \leq u$ and f_k $(k \neq j)$ is bounded. Hence, on $\Omega \cap V_j$, $|f_j|^p$ is dominated by a constant multiple of u plus a constant. By the simply-connected case, $f_j \in H^p(\Omega_j)$. Hence $f \in H^p(\Omega)$.

6.3 <u>Corollary</u> Let $1 \leq p \leq \infty$. Let $f \in H^p$ and $g \in H^\infty$. Then $fg \in H^p$.

<u>Proof</u> This is immediate from Theorems 6.1 and 6.2.

Corollary 6.3 can also be easily proved directly from the definition of H^p.

6.4 <u>Theorem</u> Let $f \in H^1(\Omega)$. Suppose that f vanishes on a set of positive measure on $\partial\Omega$. Then $f = 0$.

<u>Proof</u> The result is well-known for the unit disc D. Choose a proper subarc Γ of one of the components of $\partial\Omega$, such that f vanishes on a set of positive measure on Γ. Choose a simply-connected domain V, contained in Ω, such that $\partial V \cap \partial\Omega = \Gamma$. Choose any conformal mapping ϕ from D onto V. $|f|$ is dominated by a harmonic function u on V; so $|f \circ \phi|$ is dominated by the harmonic function $u \circ \phi$ on D. Hence $f \circ \phi \in H^1(D)$. But $f \circ \phi$ vanishes on a set of positive measure on ∂D. So $f \circ \phi = 0$ on D. Hence f vanishes on V and therefore throughout Ω.

Suppose $h \in H^1(D)$. For every $g \in A(D)$, $\int_{\partial D} h(z)g(z)dz = 0$; i.e. the measure $d\mu = h(z)dz$ on the unit circle annihilates $A(D)$. One of the most remarkable theorems in classical function theory is the F. and M. Riesz theorem, which states the converse: every (complex Borel) measure μ on ∂D which annihilates $A(D)$ is of the form $d\mu = h(z)dz$ for some $h \in H^1(D)$. We want to extend this to more general domains.

First, observe that if Ω is a bounded simply-connected domain whose boundary is an analytic Jordan curve then the F. and M. Riesz theorem goes through word for word with D replaced by Ω. To see this, choose a conformal map ϕ of Ω onto D. Since $\partial\Omega$ is an analytic curve, ϕ continues analytically across $\partial\Omega$ (an easy application of the Schwarz reflection principle). Suppose μ is a measure on $\partial\Omega$ which annihilates $A(\Omega)$. Then, for each $g \in A(D)$, $\int_{\partial D} g(z)\,d\phi^{-1}(\mu)(z) = \int_{\partial\Omega} g(\phi(\omega))\phi'(\omega)\,d\mu(\omega) = 0$ since $g(\phi(\omega))\phi'(\omega) \in A(\Omega)$. By the F. and M. Riesz theorem for the unit disc, $d\phi^{-1}(\mu)(z) = h(z)dz$ for some $h \in H^1(D)$; i.e. $d\mu(\omega) = h(\phi(\omega))\phi'(\omega)d\omega$. This is of the required form: $h(\phi(\omega)) \in H^1(\Omega)$ by the conformal invariance of H^1, and so $h(\phi(\omega))\phi'(\omega) \in H^1(\Omega)$ by Corollary 6.3.

For domains containing ∞ the situation is slightly different. If Ω is simply connected and contains ∞, then the same procedure as above shows that every measure μ on $\partial\Omega$ which annihilates $A(\Omega)$ is of the form $d\mu = h(z)dz$ where $h \in H^1(\Omega)$ and h has a zero at ∞ of order at least 2. The converse is obvious.

We can now build up the general case from these special cases.

6.5 __Theorem__ (F. and M. Riesz) Let $\Omega \in G$. Let μ be a measure on $\partial\Omega$ which annihilates $A(\Omega)$. Then μ is of the form $d\mu = h(z)dz$, where $h \in H^1(\Omega)$. If $\infty \in \Omega$ then h has a zero at ∞ of order at least 2.

__Proof__ We shall prove the result in the case when $\infty \in \Omega$: the other case is similar. Denote the components of $\partial\Omega$ by $\Gamma_1,\dots,\Gamma_n$ and write $\Omega = \bigcap_{j=1}^{n} \Omega_j$ where $\partial\Omega_j = \Gamma_j$. Write $\mu = \sum \mu_j$, where μ_j is supported on Γ_j. Define:

$$h_j(\zeta) = \int_{\Gamma_j} \frac{d\mu_j(z)}{z - \zeta}$$

for $\zeta \in \Omega_j$. Now fix j, $1 \leq j \leq n$. Let $f \in A(\Omega_j)$, $f(\infty) = 0$. Then for all $k \neq j$:

$$\int_{\Gamma_j} f(\zeta)h_k(\zeta)d\zeta = \int_{\Gamma_j} f(\zeta)\int_{\Gamma_k} \frac{d\mu_k(z)}{z - \zeta}\, d\zeta = 2\pi i\int_{\Gamma_k} f(z)d\mu_k(z)$$

by Fubini's theorem. Write:

$$\nu_j = \mu_j + \frac{1}{2\pi i}\sum_{k \neq j} h_k(z)dz$$

on Γ_j. Then $\int f\, d\nu_j = \int f\, d\mu = 0$. Hence $\dfrac{d\nu_j(z)}{z - a}$ annihilates $A(\Omega_j)$, where a is chosen inside Γ_j. By the simply-connected case, then, $d\nu_j(z) = g_j(z)dz$, where $g_j \in H^1(\Omega_j)$ and $g_j(\infty) = 0$. Let $\zeta \in \Omega$. Then:

$$\begin{aligned} h_j(\zeta) &= \int_{\Gamma_j} \frac{d\nu_j(z)}{z - \zeta} - \sum_{k \neq j} \frac{1}{2\pi i}\int_{\Gamma_k} \frac{h_k(z)dz}{z - \zeta} \\ &= \int_{\Gamma_j} \frac{g_j(z)dz}{z - \zeta} - 0 \\ &= -2\pi i g_j(\zeta). \end{aligned}$$

By the definition of ν_j, $\mu_j = h(z)dz$ on Γ_j, where $h(z) = -\frac{1}{2\pi i}\sum h_k(z)$. $h \in H^1(\Omega)$, $h(\infty) = -\frac{1}{2\pi i}\sum h_k(\infty) = 0$, and $h'(\infty) = \frac{1}{2\pi i}\int_{\partial\Omega} h(z)dz = \frac{1}{2\pi i}\int d\mu = 0$ since $1 \in A(\Omega)$.

We shall require one theorem about the canonical factorisation of functions in $H^1(D)$. We shall give a proof because the result, although well-known, does not appear explicitly in any of the standard books on Hardy spaces. If $f \in H^1(D)$ and $f \neq 0$ then Q_f denotes the outer factor of f, S_f denotes the singular factor of f, and μ_f denotes the singular measure on the unit circle giving rise to it; i.e:

$$S_f(\zeta) = \exp\left(-\int \frac{z+\zeta}{z-\zeta}\, d\mu_f(z)\right).$$

See [16], pp. 61 - 69, for background.

6.6 Theorem Let $f \in H^1(D)$, $f \not\equiv 0$. Suppose that f continues to be analytic in a neighbourhood of some point z_0 on the unit circle. Then z_0 is not in the closed support of μ_f.

Proof If $f(z_0) = 0$ then $\frac{f(z)}{z-z_0} \in H^1(D)$, and the inner factors of f and $\frac{f(z)}{z-z_0}$ are identical since $z - z_0$ is outer. So we may assume that $f(z_0) \neq 0$. Since $\log|Q_f|$ is the Poisson integral of $\log|f|$ on the unit circle, and since $\log|f|$ is continuous at z_0, $\log|Q_f|$ is continuous at z_0. Also f itself is continuous and nonzero at z_0, and the Blaschke factor of f is continuous and nonzero at z_0 since f has no zeros in a neighbourhood of z_0. So $|S_f|$ is continuous at z_0. That is, z_0 is not in the closed support of μ_f ([16], pp. 68 - 69).

§7 The Dual Problem

In chapter I we studied the extremal problem:

(1) "$g'(\infty) = \sup\{|g'(\infty)| : g$ is analytic on Ω, $|g| < 1$, $g(\infty) = 0\}$".

It was merely for convenience that we selected ∞ as a special point: indeed, it is clear that if Ω is any domain in S^2 and $\zeta \in \Omega$ then the problem:

(2) "$g'(\zeta) = \sup\{|g'(\zeta)| : g$ is analytic on Ω, $|g| < 1$, $g(\zeta) = 0\}$"

is converted into a problem of the form (1) simply by rotating S^2

by a linear fractional transformation. Hence, for instance, (2) always has a unique solution. However, the effect of rotating S^2 on the other problems we are about to study is not so trivial, and so we dare not initially select ∞ as our point of reference.

Let $\Omega \in G$, and let $\zeta \in \Omega$, $\zeta \neq \infty$. (The case $\zeta = \infty$ will be studied in §10.) We shall use the following notation. H^p denotes $H^p(\Omega)$. H^{1+} denotes the linear span, in L^1, of H^1 and the functions $\dfrac{1}{z-\zeta}$ and $\dfrac{1}{(z-\zeta)^2}$. Thus H^{1+} is a closed linear subspace of L^1. If $\infty \in \Omega$, then H_0^1 denotes the subspace of H^1 consisting of those functions in H^1 which have a zero at ∞ of order at least 2, and H_0^{1+} denotes the subspace of H^{1+} consisting of those functions in H^{1+} which have a zero at ∞ of order at least 2. H^{2+} denotes the linear span, in L^2, of H^2 and the function $\dfrac{1}{z-\zeta}$. Thus H^{2+} is a closed linear subspace of L^2. If $\infty \in \Omega$, then H_0^2 denotes the subspace of H^2 consisting of those functions in H^2 which vanish at ∞, and H_0^{2+} denotes the subspace of H^{2+} consisting of those functions in H^{2+} which vanish at ∞.

7.1 Theorem Let Ω be a bounded domain in S^2 whose boundary is the union of n pairwise disjoint analytic Jordan curves $\Gamma_1, \ldots, \Gamma_n$. Let $\zeta \in \Omega$. Let f be the (unique) solution of the extremal problem:

$$\text{"}g'(\zeta) = \sup\{|g'(\zeta)| : g \text{ is analytic on } \Omega,\ |g| < 1,\ g(\zeta) = 0\}\text{"}.$$

Then:

(a) f is analytic across $\partial\Omega$;

(b) $|f| = 1$ on $\partial\Omega$: in fact,

(c) f maps each Γ_j homeomorphically onto the unit circle, and the variation of $\arg f$ round each Γ_j is 2π;

(d) f takes each value $\omega \in D$ precisely n times, counting multiplicities.

There is a unique function $\psi \in H^{1+}$ such that:

(e) $\displaystyle\int_{\partial\Omega} g(z)\psi(z)dz = g'(\zeta)$ for all $g \in A(\Omega)$ and

(f) $\displaystyle\int_{\partial\Omega} |\psi| ds = f'(\zeta)$.

Further:

(g) ψ is analytic across $\partial\Omega$;

(h) ψ has no zeros on $\overline{\Omega}$;

(i) $f(z)\psi(z)dz \geq 0$ on $\partial\Omega$;

(j) the expansion of ψ near ζ has the form:

$$\psi(z) = \frac{1}{2\pi i(z-\zeta)^2} + a_0 + a_1(z-\zeta) + \ldots\ldots$$

<u>Proof</u> Note first that if ψ is any function in H^{1+} such that (e) is satisfied for all $g \in A(\Omega)$, then (e) is satisfied for all $g \in H^\infty$, since both sides of (e) are weak-* continuous functionals on H^∞ and $A(\Omega)$ is weak-* dense in H^∞.

$g \to g'(\zeta)$ defines a continuous linear functional on $A(\Omega)$ of norm at most $f'(\zeta)$: for if $g \in A(\Omega)$ and $|g| < 1$ then $|g'(\zeta)| \leq f'(\zeta)$ by the remark following the definition of analytic capacity. This linear functional extends, by the Hahn-Banach theorem, to a continuous linear functional on the whole of $C(\partial\Omega)$ with the same norm, and by the Riesz representation theorem that functional is represented by some measure σ on $\partial\Omega$. Then $\|\sigma\| \leq f'(\zeta)$ and $\int g\, d\sigma = g'(\zeta)$ $(g \in A(\Omega))$. But also $\displaystyle\frac{1}{2\pi i}\int_{\partial\Omega}\frac{g(z)dz}{(z-\zeta)^2}$ $= g'(\zeta)$ $(g \in A(\Omega))$. So the measure $d\mu = \dfrac{dz}{2\pi i(z-\zeta)^2} - d\sigma$ annihilates $A(\Omega)$. By the F. and M. Riesz theorem (Theorem 6.5) there is an element $h \in H^1$ such that $d\mu = h(z)dz$. So $d\sigma =$ $(1/2\pi i(z-\zeta)^2 - h(z))dz = \psi(z)dz$, say, where $\psi \in H^{1+}$. If $g \in A(\Omega)$

then $\int_{\partial\Omega} g(z)\psi(z)dz = \int g\,d\sigma = g'(\zeta)$. Hence (e). By the first paragraph of the proof, $f'(\zeta) = \int_{\partial\Omega} f(z)\psi(z)dz$; so that $f'(\zeta) \leq \int_{\partial\Omega} |\psi|\,ds = \int |d\sigma| = \|\sigma\| \leq f'(\zeta)$. Hence (f).

From now on, ψ denotes <u>any</u> function in H^{1+} satisfying (e) and (f). The functions 1 and z are in $A(\Omega)$, so that, by (e), $\int_{\partial\Omega} \psi(z)dz = 0$ and $\int_{\partial\Omega} z\psi(z)dz = 1$. Hence (j). $\int_{\partial\Omega} |\psi|\,ds = f'(\zeta) = \int_{\partial\Omega} f(z)\psi(z)dz$, and $|f| \leq 1$. Moreover, ψ cannot vanish on a set of positive measure on $\partial\Omega$, by Theorem 6.4, since $(z-\zeta)^2\psi(z) \in H^1$. Therefore:

(3) $$f(z)\psi(z)dz \geq 0 \quad \text{a.e. on } \partial\Omega;$$

(4) $$|f| = 1 \quad \text{a.e. on } \partial\Omega.$$

Let $1 \leq j \leq n$. Γ_j is an analytic curve. So there is a number $r < 1$ and a conformal map τ, defined on $\{\omega : r < |\omega| < \frac{1}{r}\}$, such that $\tau(\partial D) = \Gamma_j$ and $\tau\{\omega : r < |\omega| < 1\} \subset \Omega$. Write $h(\omega) = f(\tau(\omega))\psi(\tau(\omega))$ for $r < |\omega| < 1$. Clearly $h \in H^1\{\omega : r < |\omega| < 1\}$, and, by (3), $i\omega h(\omega)\tau'(\omega) \geq 0$ on ∂D. By the Schwarz reflection principle, $i\omega h(\omega)\tau'(\omega)$ (and hence $h(\omega)$) continues analytically to $\{\omega : r < |\omega| < \frac{1}{r}\}$. Thus:

(5) $$f\psi \text{ is analytic on } \partial\Omega.$$

Now, at ζ, f has a simple zero and ψ has a double pole; and elsewhere in Ω, $f\psi$ is analytic. By the argument principle, (3) therefore implies that $f\psi$ has, in addition to its simple pole at ζ:

(6) $$\left[\begin{array}{ll} \underline{\text{either}} & \text{precisely } n-1 \text{ zeros in } \Omega \text{ and none on } \partial\Omega \\ \underline{\text{or}} & \text{fewer than } n-1 \text{ zeros in } \Omega. \end{array}\right.$$

Let Γ be a proper open subarc of one of the Γ_j. Choose a simply-connected domain V, contained in Ω, such that $\partial V \cap \partial\Omega = \overline{\Gamma}$, $\zeta \notin V$, and $f\psi$ has no zeros on V. Choose a conformal map ϕ of D onto V. There is an open subarc J of the unit circle such that ϕ is analytic across J and $\phi(J) = \Gamma$. Write $\tilde{f} = f \circ \phi$ and $\tilde{\psi} = \psi \circ \phi$. Then $\tilde{f} \in H^\infty(D)$, $\tilde{\psi} \in H^1(D)$, $\tilde{f}$ and $\tilde{\psi}$ have no zeros in D, and $\tilde{f}\tilde{\psi}$ is analytic on J. By Theorem 6.6, the measure $\mu_{\tilde{f}} + \mu_{\tilde{\psi}}$ giving rise to the singular factor of $\tilde{f}\tilde{\psi}$ is supported off J. But $\mu_{\tilde{f}} \geq 0$ and $\mu_{\tilde{\psi}} \geq 0$; so $\mu_{\tilde{f}}$ is supported off J. Hence $S_{\tilde{f}}$ is analytic across J. But also:

$$Q_{\tilde{f}}(\omega) = \exp\left(\frac{1}{2\pi}\int_{\partial D} \frac{z+\omega}{z-\omega} \log|\tilde{f}(z)|\,ds\right)$$

is analytic across J since $\log|\tilde{f}| = 0$ a.e. on J by (4). So $\tilde{f} = S_{\tilde{f}}Q_{\tilde{f}}$ is analytic across J; i.e. f is analytic across Γ. Hence (a). (b) follows from (a) and (4). (g) holds because $f\psi$ and f are analytic across $\partial\Omega$ and f does not vanish on $\partial\Omega$. (i) follows from (3).

Let $1 \leq j \leq n$. $\int_{\Gamma_j} d(\arg f) = \int_{\Gamma_j} \frac{\partial \log|f|}{\partial n}\,ds > 0$ since $|f| = 1$ on Γ_j, $|f| < 1$ in Ω, and $\partial \log|f|/\partial n$ cannot vanish on an arc in Γ_j. Hence $\frac{1}{2\pi}\int_{\Gamma_j} d(\arg f) \geq 1$, the left hand side being an integer. But $\sum_{j=1}^{n} \frac{1}{2\pi}\int_{\Gamma_j} d(\arg f)$ is the number of zeros of f in Ω, which is at most n by (6). Hence, for each j:

$$\int_{\Gamma_j} d(\arg f) = 2\pi. \tag{7}$$

$\frac{\partial(\arg f)}{\partial s} > 0$; this fact, together with (7), gives (c). The number of zeros of f in Ω is $\sum_{j=1}^{n} \frac{1}{2\pi}\int_{\Gamma_j} d(\arg f)$, which we know is just n. By the argument principle, f takes each value $\omega \in D$ n times: hence (d). (6) now says that ψ has no zeros on Ω or $\partial\Omega$. Hence (h).

Finally we show uniqueness of ψ. Suppose that ψ_1 and

ψ_2 both satisfy (e) and (f). We have shown that they satisfy (g), (h), (i) and (j) also. $\frac{\psi_1}{\psi_2}$ is analytic on $\overline{\Omega}$ by (g) and (h), and real on $\partial\Omega$ by (i). Hence $\frac{\psi_1}{\psi_2}$ is constant. Its constant value is 1 by (j). So $\psi_1 = \psi_2$.

The above proof is essentially the same as the one given by Garnett ([13], pp. 18 - 21). We have given more detail.

Let $h \in H^{1+}$. Suppose that the coefficient of $\frac{1}{(z-\zeta)^2}$ in the expansion of h about ζ is $\frac{1}{2\pi i}$. We know then that $\int_{\partial\Omega} g(z)h(z)dz = g'(\zeta)$ whenever $g \in H^\infty(\Omega)$, $\|g\| \leqslant 1$, and $g(\zeta) = 0$. In particular, $f'(\zeta) = \int_{\partial\Omega} f(z)h(z)dz \leqslant \int_{\partial\Omega} |h|ds$, where f is as in Theorem 7.1. Thus Theorem 7.1 shows that ψ solves the extremal problem:

(8) $$\text{"}\int_{\partial\Omega} |h|ds = \inf\{\int_{\partial\Omega} |h|ds : h \in H^{1+},\ h(z) = \frac{1}{2\pi i(z-\zeta)^2} + \ldots\ldots\}\text{"}.$$

The extremal problems (2) and (8) are called "dual problems" in view of the connections between them which are brought out by the proof of Theorem 7.1. See [6], chapter 8, for an excellent exposition of a more general setting of this concept.

If $\infty \in \Omega$, then the statement of Theorem 7.1 needs slight modifications, due both to the special properties of ∞ and to the unfortunate convention of the direction of contour integration. The required changes are: the variation of $\arg f$ round each Γ_j is -2π, instead of 2π; $\psi \in H_0^{1+}$, instead of H^{1+}; $\int_{\partial\Omega} g(z)\psi(z)dz = -g'(\zeta)$, instead of $g'(\zeta)$; ψ has no zeros on $\overline{\Omega}$ except for its statutory double zero at ∞; and $f(z)\psi(z)dz \leqslant 0$ on $\partial\Omega$, instead of $\geqslant 0$. The proof is practically unchanged.

Some of the properties of the extremal function f given by Theorem 7.1 carry over to more general domains. The following corollary is an example.

7.2 <u>Corollary</u> Let Ω be a domain in S^2. Suppose that $S^2 - \Omega$ consists of n components, none of which is a singleton. Let $\zeta \in \Omega$ and let f be the solution of the extremal problem (2). Then:

(a) f takes each value $\omega \in D$ precisely n times, counting multiplicities;

(b) $|f|$ extends continuously to $\partial\Omega$ and $|f| = 1$ on $\partial\Omega$;

(c) if one component of $\partial\Omega$ is a Jordan curve Γ then f extends to be a homeomorphism of Γ onto ∂D;

(d) if one component of $\partial\Omega$ is a Jordan curve containing an analytic arc Γ then f continues analytically across Γ.

<u>Proof</u> Map Ω conformally onto a bounded domain of type G. The results follow from Theorem 7.1, the conformal invariance of (2) and the boundary properties of conformal mapping.

§8 <u>The Szegö Kernel Function</u>

In the last section we looked at a pair of extremal problems, one in H^∞ and one in H^1. In this section we examine a related pair of problems in H^2. As in the last section, Ω is a domain in S^2 whose boundary is the union of n pairwise disjoint analytic Jordan curves in $\underset{\sim}{C}$, $\zeta \in \Omega$, $\zeta \neq \infty$, and f and ψ are the solutions of the extremal problems (2) and (8) respectively of the last section.

Suppose first that Ω is bounded. By Theorem 7.1 (c) and (i), the variation of $\arg \psi$ is -4π round the outer boundary of $\bar{\Omega}$ and 0 round each of the other components of $\partial\Omega$. Also ψ has no zeros in Ω and its only pole is a double one at ζ. Therefore ψ has a single-valued square root in Ω. We define:

$$L_\zeta(z) = \left(\frac{i\psi(z)}{2\pi}\right)^{\frac{1}{2}}$$

$$K_\zeta(z) = f(z)\left(\frac{i\psi(z)}{2\pi}\right)^{\frac{1}{2}},$$

so that $f = K_\zeta/L_\zeta$. The sign of the square root is chosen so that the residue of L_ζ at ζ is positive. K_ζ and L_ζ are called respectively the <u>Szegö kernel</u> and the <u>Szegö co-kernel</u> of Ω at ζ. We shall also use the notation $K(z,\zeta)$ for $K_\zeta(z)$ and $L(z,\zeta)$ for $L_\zeta(z)$. Theorem 7.1 tells us immediately that K_ζ is analytic on $\overline{\Omega}$, that $K(\zeta,\zeta) = \frac{f'(\zeta)}{2\pi}$, that K_ζ has $n-1$ zeros in Ω and none on $\partial\Omega$, that L_ζ is analytic on $\overline{\Omega}$ except for a simple pole at ζ of residue $\frac{1}{2\pi}$, and that L_ζ never vanishes. Theorem 7.1 also gives the important relation:

(1) $$\overline{K}_\zeta ds = \frac{1}{i}L_\zeta dz \quad \text{round} \quad \partial\Omega,$$

because $\frac{1}{i}\frac{L_\zeta(z)}{\overline{K_\zeta(z)}}\frac{dz}{ds}$ is positive by (i) and has modulus 1 by (b).

Let $h \in H^2$. Then $\int_{\partial\Omega} h(z)\overline{K_\zeta(z)}ds = \frac{1}{i}\int_{\partial\Omega} h(z)L_\zeta(z)dz = h(\zeta)$ by the residue theorem. So K_ζ is the element of H^2 which represents the functional on H^2 given by evaluation at ζ. Equivalently, K_ζ is the element of H^2 orthogonal to the hyperplane $\{h \in H^2 : h(\zeta) = 0\}$ and normalised by the relation $\|K_\zeta\|^2 = K(\zeta,\zeta)$. Equivalently, $\frac{K_\zeta}{K(\zeta,\zeta)^{\frac{1}{2}}}$ is the solution of the extremal problem:

$$\text{"}g(\zeta) = \sup\{|g(\zeta)| : g \in H^2, \|g\|_2 \leq 1\}\text{".}$$

If $\{u_n\}$ is an orthonormal basis for H^2, then, by Parseval's theorem, $K_\zeta = \sum_n u_n\overline{(u_n,K_\zeta)} = \sum_n u_n\overline{u_n(\zeta)}$. This can be written more symmetrically as:

$$(2) \qquad K(z,\zeta) = \sum_n u_n(z)\overline{u_n(\zeta)}.$$

Hence $K(z,\zeta) = \overline{K(\zeta,z)}$. This shows that $K(z,\zeta)$ is analytic in $\overline{\zeta}$ as well as in z. Since $K(z,\zeta) = (K_\zeta,K_z)$, $|K(z,\zeta)|^2 \leq K(z,z)K(\zeta,\zeta)$ by the Cauchy-Schwarz inequality.

The co-kernel fulfils an analogous rôle in H^{2+}. Denote the residue at ζ of a function $h \in H^{2+}$ by res(h). If $h \in H^{2+}$ then $\int_{\partial\Omega} h(z)\overline{L_\zeta(z)}ds = \frac{1}{i}\int_{\partial\Omega} h(z)K_\zeta(z)dz = 2\pi K(\zeta,\zeta)\mathrm{res}(h)$. So $\dfrac{L_\zeta}{2\pi K(\zeta,\zeta)}$ is the element of H^{2+} which represents the functional "res". Equivalently, L_ζ is the element of H^{2+} orthogonal to H^2 and normalised by the constraint $\mathrm{res}(L_\zeta) = \frac{1}{2\pi}$. Equivalently, L_ζ is the solution of the extremal problem:

$$\|g\|_2 = \inf\{\|g\|_2 : g \in H^{2+}, \mathrm{res}(g) = \tfrac{1}{2\pi}\}.$$

If $\{u_n\}$ is an orthonormal basis for H^{2+} then:

$$(3) \qquad \frac{L(z,\zeta)}{2\pi K(\zeta,\zeta)} = \sum_n u_n(z)\overline{\mathrm{res}(u_n)}.$$

If $z, \zeta \in \Omega$ then:

$$\begin{aligned} L(z,\zeta) + L(\zeta,z) &= \frac{1}{i}\int_{\partial\Omega} L(\eta,\zeta)L(\eta,z)d\eta && \text{by the residue theorem} \\ &= \overline{\frac{1}{i}\int_{\partial\Omega} K(\eta,\zeta)K(\eta,z)d\eta} && \text{by (1)} \\ &= 0 && \text{by Cauchy's theorem.} \end{aligned}$$

Hence $L(z,\zeta) = -L(\zeta,z)$. This shows that $L(z,\zeta)$ is analytic in ζ as well as in z.

Incidentally, the only properties of K_ζ and L_ζ which have been assumed in the last two paragraphs are that $K_\zeta \in H^2$,

$L_\zeta \in H^{2+}$, $\operatorname{res}(L_\zeta) = 1/2\pi$, and $\overline{K_\zeta}ds = (1/i)L_\zeta dz$. We have therefore shown that K_ζ and L_ζ constitute the only pair of functions with these properties.

We can express L in terms of K, using the residue theorem and (1), as follows:

$$
\begin{aligned}
L(z,\zeta) &= \frac{1}{2\pi(z-\zeta)} + \frac{1}{2\pi i}\int_{\partial\Omega} L(\eta,\zeta)\,\frac{d\eta}{\eta - z} \\
&= \frac{1}{2\pi(z-\zeta)} + \frac{1}{2\pi}\int_{\partial\Omega} \frac{K(\zeta,\eta)}{\eta - z}|d\eta|. \qquad (4)
\end{aligned}
$$

Again, if $\infty \in \Omega$ then slight changes are necessary. In that case, $\arg\psi$ has variation 0 round each component of $\partial\Omega$. ψ has a double pole at ζ and a double zero at ∞, and is otherwise free from zeros and poles. Hence ψ has a square root as before. H^2 and H^{2+} must be replaced by H_0^2 and H_0^{2+} respectively throughout. K_ζ has n zeros, including at least one at ∞. L_ζ has a simple zero at ∞ and otherwise does not vanish. If we define $K(z,\infty) = 0$ and $L(z,\infty) = 0$ then K and L remain analytic wherever they are defined, and the relations $K(z,\zeta) = \overline{K(\zeta,z)}$ and $L(z,\zeta) = -L(\zeta,z)$ are preserved. The relation (1) acquires a $-$ sign. Everything else is unchanged and only minor modifications are needed in the proofs.

The next result shows how the kernels behave under conformal mapping. We shall use the following observation: if ϕ is any conformal mapping of a domain $\Omega^{(2)} \subset S^2$ onto a domain $\Omega^{(1)} \subset S^2$, then ϕ' has a single-valued square root on $\Omega^{(2)}$, since if Γ is any closed curve in $\Omega^{(2)}$ and neither Γ nor $\phi(\Gamma)$ passes through ∞ then the variation of $\arg\phi'$ round Γ is -4π or 0 according as ϕ does or does not reverse the sense of Γ.

8.1 <u>Theorem</u> Let $\Omega^{(1)}$ and $\Omega^{(2)}$ be bounded domains of type G, and let ϕ map $\Omega^{(2)}$ conformally onto $\Omega^{(1)}$. Then:

(a) whenever $\{u_n\}$ is an orthonormal basis for $H^2(\Omega^{(1)})$ (respectively, $H^{2+}(\Omega^{(1)})$), $\{\phi'(z)^{\frac{1}{2}} u_n(\phi(z))\}$ is an orthonormal basis for $H^2(\Omega^{(2)})$ (respectively, $H^{2+}(\Omega^{(2)})$);

(b) if $K^{(1)}$ and $K^{(2)}$ are the Szegö kernels, and $L^{(1)}$ and $L^{(2)}$ the co-kernels, of $\Omega^{(1)}$ and $\Omega^{(2)}$ respectively then:

$$K^{(2)}(z,\zeta) = \phi'(z)^{\frac{1}{2}} \overline{\phi'(\zeta)^{\frac{1}{2}}} K^{(1)}(\phi(z),\phi(\zeta)) \qquad (z, \zeta \in \Omega^{(2)})$$

$$L^{(2)}(z,\zeta) = \phi'(z)^{\frac{1}{2}} \phi'(\zeta)^{\frac{1}{2}} L^{(1)}(\phi(z),\phi(\zeta)) \qquad (z, \zeta \in \Omega^{(2)}, z \neq \zeta).$$

<u>Proof</u> It will not matter which square root of ϕ' we use as long as we always use the same one. For each $u \in H^{2+}(\Omega^{(1)})$, define a function Tu on $\Omega^{(2)}$ by $(Tu)(z) = \phi'(z)^{\frac{1}{2}} u(\phi(z))$. It is easy to verify that T is a unitary map of $H^{2+}(\Omega^{(1)})$ onto $H^{2+}(\Omega^{(2)})$ and that T maps $H^2(\Omega^{(1)})$ onto $H^2(\Omega^{(2)})$. Hence (a). (b) is an immediate consequence of (a) and the formulae (2) and (3).

If for $i = 1$ or 2 $\infty \in \Omega^{(i)}$ then the result still holds if $H^2(\Omega^{(i)})$ and $H^{2+}(\Omega^{(i)})$ are replaced by $H_0^2(\Omega^{(i)})$ and $H_0^{2+}(\Omega^{(i)})$ respectively.

§9 <u>Extension to Arbitrary Domains</u>

In §7 and §8 we considered only domains of type G. In this section we show how to define the Szegö kernel and co-kernel of an arbitrary domain in S^2.

9.1 <u>Proposition</u> Let Ω be a domain in S^2, $\Omega \neq S^2$. Then Ω is the union of a sequence $\{\Omega_n\}$ in G such that $\overline{\Omega}_n \subset \Omega_{n+1}$ for each n.

Proof By an easy compactness argument, it is easy to see that Ω is the union of an increasing sequence of finitely connected domains: so we may assume that Ω has finite connectivity n. Ω is the intersection of n simply-connected domains, none of which is the whole of S^2, and whose boundaries are pairwise disjoint; and so it is sufficient to show the result when Ω is simply-connected. Choose any conformal mapping ϕ of D onto Ω; if $\infty \in \Omega$ then choose ϕ so that $\phi(0) = \infty$. Then the required sequence is $\{\phi(D(0;\frac{n}{n+1}))\}$.

9.2 Lemma Let $\Omega \in G$. Let z and ζ be distinct finite points of Ω. Write $r = d(z, \partial\Omega)$. Then the Szegö kernel K and the co-kernel L of Ω satisfy the bounds:

(a) $K(z,z) \leq \frac{1}{2\pi r}$;

(b) $\left|L(z,\zeta)^2 - \frac{1}{4\pi^2(z-\zeta)^2}\right| \leq \frac{K(\zeta,\zeta)}{2\pi r}$.

Proof (a) $K(z,z) = \frac{1}{2\pi}\sup\{|g'(z)| : g$ is analytic on Ω and $|g| < 1\}$. If g is analytic and bounded by 1 on Ω then in particular g is analytic and bounded by 1 on $D(z;r)$, so that $|g'(z)| \leq \frac{1}{r}$ by Schwarz's lemma.

(b) Theorem 7.1 (j) says that, for η near ζ, $L(\eta,\zeta)^2$ is of the form $\frac{1}{4\pi^2(\eta-\zeta)^2} + a_0 + a_1(\eta-\zeta) + \dots$. Hence by the residue theorem:

$$\begin{aligned}\left|L(z,\zeta)^2 - \frac{1}{4\pi^2(z-\zeta)^2}\right| &= \left|\frac{1}{2\pi i}\int_{\partial\Omega}\frac{L(\eta,\zeta)^2 d\eta}{\eta - z}\right| \\ &\leq \frac{1}{2\pi r}\int_{\partial\Omega}|L(\eta,\zeta)|^2 d\eta \\ &= \frac{K(\zeta,\zeta)}{2\pi r}.\end{aligned}$$

9.3 <u>Proposition</u> Let V be a domain in $\underset{\sim}{C}^2$ which contains the point $(z,\bar{z})$ for some $z \in \underset{\sim}{C}$. Let f be an analytic function of two variables on V with the property that $f(z,\bar{z}) = 0$ whenever $(z,\bar{z}) \in V$. Then $f = 0$.

<u>Proof</u> We may suppose that $(0,0) \in V$. Near $(0,0)$, $f(z,\zeta) = \sum_{j,k=0}^{\infty} a_{jk} z^j \zeta^k$ by Taylor's theorem. By hypothesis, for all sufficiently small positive r, $0 = f(re^{i\theta}, re^{-i\theta}) = \sum_{j,k=0}^{\infty} a_{jk} r^{j+k} e^{i(j-k)\theta}$. Equating coefficients of powers of r, we find that for all $n \geqslant 0$ and for all $\theta \in \underset{\sim}{R}$, $\sum_{j+k=n} a_{jk} e^{i(j-k)\theta} = 0$. Since the sequence of functions $\{e^{mi\theta}\}$ is linearly independent, $a_{jk} = 0$ for all j and k. So $f = 0$.

We now show how to define the Szegö kernel and co-kernel of an arbitrary domain in S^2. Let Ω be a domain in S^2, $\Omega \neq S^2$. Ω is the union of an increasing sequence $\{\Omega_n\}$ in G by Proposition 9.1. For each n let K_n be the Szegö kernel of Ω_n. By Lemma 9.2 (a), $\{K_n(z,\zeta)\}$, as a sequence of analytic functions of the two variables z and $\bar{\zeta}$, is normal. The result of Theorem 2.4, rotated by a linear fractional transformation, says that for each point $z \in \Omega$ $K_n(z,z)$ converges as $n \to \infty$. If $K^{(1)}$ and $K^{(2)}$ are cluster points of the sequence $\{K_n\}$ then $K^{(1)}(z,z) = \lim_{n\to\infty} K_n(z,z) = K^{(2)}(z,z)$ for each finite point $z \in \Omega$, so that $K^{(1)} - K^{(2)} = 0$ by Proposition 9.3. Hence $\{K_n\}$ has a unique cluster point K, and therefore converges uniformly on compact sets to K. (The theory of normal families for functions of several variables is similar to that for functions of one variable: see [17], p. 26.) The same reasoning shows that K is independent of the choice of sequence $\{\Omega_n\}$. We shall call K the <u>Szegö kernel</u> of Ω. The same reasoning now shows also that if $\{\Omega_n\}$ is <u>any</u> increasing sequence of domains with union Ω $(\Omega \neq S^2)$ then the Szegö kernels of Ω_n converge uniformly on compact sets to the Szegö kernel of Ω. If K is the Szegö kernel of a domain Ω and $z \in \Omega$ then $K(z,z) = \frac{1}{2\pi} \sup\{|f'(z)| : f$ is analytic on Ω and $|f| < 1\}$, since this holds for all $\Omega \in G$. Hence if $\gamma(S^2 - \Omega) = 0$ then $K(z,z) = 0$ for all $z \in \Omega$ and so $K = 0$. We define the Szegö kernel of S^2 to be 0 also.

To define the co-kernel of an arbitrary domain Ω we must distinguish two cases. Suppose first that $\gamma(S^2 - \Omega) > 0$. We define $L(z,\zeta) = L_\zeta(z) = \dfrac{K_\zeta(z)}{f(z)}$ in the notation of §8. Every zero of f, except the one at ζ, is cancelled by a zero of K_ζ (since this holds if $\Omega \in G$), and so L_ζ is analytic on Ω except for a simple pole at ζ, which clearly has residue $\frac{1}{2\pi}$. If $\{\Omega_n\}$ is any increasing sequence of domains with union Ω then the co-kernels of Ω_n converge uniformly on compact sets to the co-kernel of Ω, since the same holds for the kernels.

Suppose now that $\gamma(S^2 - \Omega) = 0$. Define the co-kernel of Ω to be the function L given by $L(z,\zeta) = \dfrac{1}{2\pi(z-\zeta)}$. Lemma 9.2 (b) now holds for every domain in S^2: for domains whose complement has positive analytic capacity this follows from the case when $\Omega \in G$, and for domains whose complement has zero analytic capacity it holds by definition. If $\{\Omega_n\}$ is any increasing sequence of domains with union Ω and L_n is the co-kernel of Ω_n, then $L_n^2 \to L^2$ uniformly on compact sets by Lemma 9.2 (b). Hence the only possible cluster points of $\{L_n\}$ are L and $-L$. In fact the behaviour of $L_n(z,\zeta)$ and $L(z,\zeta)$ when z is near ζ shows that $-L$ is not a cluster point of $\{L_n\}$. So $L_n \to L$ uniformly on compact sets.

We have therefore proved the following theorem.

9.4 <u>Theorem</u> The Szegö kernel and co-kernel can be defined for every domain $\Omega \subset S^2$ in such a way that:

(a) the definitions coincide with the existing meanings if $\Omega \in G$;

(b) the Szegö kernels (respectively, co-kernels) of an increasing sequence of domains converge to the Szegö kernel (respectively, co-kernel) of its union, uniformly on compact sets. If $\gamma(S^2 - \Omega) = 0$ then the Szegö kernel of Ω is 0, and the co-kernel L is given by the formula $L(z,\zeta) = \dfrac{1}{2\pi(z-\zeta)}$.

Several of the properties of the Szegö kernel K and co-kernel L of a domain Ω which we have proved in the case when $\Omega \in G$ now follow for arbitrary domains $\Omega \subset S^2$ by Theorem 9.4. For example: $K(z,\zeta) = \overline{K(\zeta,z)}$, $L(z,\zeta) = -L(\zeta,z)$, L never vanishes, and if $\infty \in \Omega$ then $K(\infty,\zeta) = 0$ and $L(\infty,\zeta) = 0$. The generalisation of Theorem 8.1 is sufficiently important to be stated separately.

9.5 <u>Theorem</u> Let $\Omega^{(1)}$ and $\Omega^{(2)}$ be domains in S^2, and let ϕ map $\Omega^{(2)}$ conformally onto $\Omega^{(1)}$. Let $K^{(1)}$ and $K^{(2)}$ be the Szegö kernels, and $L^{(1)}$ and $L^{(2)}$ the co-kernels, of $\Omega^{(1)}$ and $\Omega^{(2)}$ respectively. Then:

$$K^{(2)}(z,\zeta) = \phi'(z)^{\frac{1}{2}}\, \overline{\phi'(\zeta)^{\frac{1}{2}}}\, K^{(1)}(\phi(z),\phi(\zeta)) \qquad (z,\zeta \in \Omega^{(2)});$$
$$L^{(2)}(z,\zeta) = \phi'(z)^{\frac{1}{2}}\, \phi'(\zeta)^{\frac{1}{2}}\, L^{(1)}(\phi(z),\phi(\zeta)) \qquad (z,\zeta \in \Omega^{(2)},\ z \neq \zeta).$$

<u>Proof</u> If $\Omega^{(2)} = S^2$ then $\Omega^{(1)} = S^2$ and the result is trivial. Otherwise, by Proposition 9.1 there is an increasing sequence $\{\Omega_n^{(2)}\}$ in G such that $\overline{\Omega_n^{(2)}} \subset \Omega^{(2)}$ for each n and $\cup\, \Omega_n^{(2)} = \Omega^{(2)}$. ϕ maps $\Omega_n^{(2)}$ conformally onto a domain $\Omega_n^{(1)}$ bounded by finitely many pairwise disjoint analytic Jordan curves, which we

may suppose lie in $\underset{\sim}{C}$, and $\{\Omega_n^{(1)}\}$ is an increasing sequence whose union is $\Omega^{(1)}$. By Theorem 8.1 and the remark after it, the required identities hold with the kernels and co-kernels of $\Omega^{(2)}$ and $\Omega^{(1)}$ replaced by the corresponding kernels and co-kernels of $\Omega_n^{(2)}$ and $\Omega_n^{(1)}$. Let $n \to \infty$. The result follows by Theorem 9.4.

§10 The Special Rôle of ∞

In the last three sections we have used a finite point $\zeta \in \Omega$ as reference point. If $\infty \in \Omega$ then the theory can be developed analogously with ∞ as reference point: but because the analogy is not transparent, and because it has been further confused in the literature by poor notation, it is as well to examine it closely.

Let E be a compact plane set. Write $\Omega = \Omega(E)$ and assume that $\Omega \in G$. The unique solution of the extremal problem:

$$"g(\infty) = \sup\{|g'(\infty)| : g \text{ is analytic on } \Omega, |g| < 1\}"$$

is, of course, the Ahlfors function f_E of E. A procedure exactly analogous to the proof of Theorem 7.1 shows the existence of a unique solution, ψ_E, of the extremal problem:

$$"\|h\|_1 = \inf\{\|h\|_1 : h \in H^1(\Omega), h(\infty) = \frac{1}{2\pi i}\}",$$

and shows that f_E and ψ_E are analytic across $\partial\Omega$, that the variations of $\arg f_E$ and $\arg \psi_E$ are -2π and 0 respectively round each component of $\partial\Omega$, that f_E maps each component of $\partial\Omega$ homeomorphically onto the unit circle, that f_E has n zeros on Ω and ψ_E has none on $\overline{\Omega}$, and that $f_E(z)\psi_E(z)dz \geq 0$ on $\partial\Omega$. ψ_E is called the <u>Garabedian function</u> of E. The Garabedian function is the analogue of the function ψ introduced in §7; observe that the double zero at ∞ and the double pole at ζ have, in some sense, "cancelled each other out", so that the Garabedian function has the finite value $\frac{1}{2\pi i}$ at ∞.

The method of §8 shows that ψ_E has a square root in Ω, and that the function:

$$K^+(z,\infty) = \frac{1}{\gamma(E)}\left(\frac{i\psi_E(z)}{2\pi}\right)^{\frac{1}{2}}$$

is the (unique) element of H^2 representing evaluation at ∞ in H^2. (We take that branch of the square root which is positive at ∞.) $K^+(z,\infty)$ is called the Szegö kernel of Ω at ∞, though we shall see that this is misleading. Since $f_E(z)\psi_E(z)dz \geqslant 0$ on $\partial\Omega$, we have:

(1) $$\overline{K^+(z,\infty)}\,\overline{f_E(z)}ds = \frac{K^+(z,\infty)}{i}dz.$$

Hence if $g \in H_0^2$ then:

$$\begin{aligned}\gamma(E)\int_{\partial\Omega} g(z)\overline{K^+(z,\infty)}\,\overline{f_E(z)}ds &= \frac{\gamma(E)}{i}\int_{\partial\Omega} g(z)K^+(z,\infty)dz \\ &= 2\pi\gamma(E)K^+(\infty,\infty)g'(\infty) \\ &= g'(\infty).\end{aligned}$$

That is, $\gamma(E)K^+(z,\infty)f_E(z)$ is the unique element of H_0^2 representing derivation at ∞.

The best way to see the analogy between the kernels at ∞ and the kernels at a finite point ζ is to examine the effect of a conformal mapping which takes ∞ to ζ. Let $\Omega^{(2)}$ be a domain in S^2, containing ∞, whose boundary is the union of n disjoint analytic Jordan curves. Let $K^{(2)+}(z,\infty)$ be the Szegö kernel of $\Omega^{(2)}$ at ∞. Let $E = S^2 - \Omega^{(2)}$. Suppose that $\Omega^{(1)}$ is a domain whose boundary is again the union of n pairwise disjoint analytic Jordan curves in $\underset{\sim}{C}$, and suppose that ϕ is a conformal mapping of $\Omega^{(2)}$ onto $\Omega^{(1)}$ which takes ∞ to some point $\zeta \in \underset{\sim}{C}$. For simplicity we assume that $\Omega^{(1)}$ is bounded: if $\infty \in \Omega^{(1)}$ then the analysis is identical except that $H^2(\Omega^{(1)})$ is replaced by $H^2_0(\Omega^{(1)})$ and $H^{2+}(\Omega^{(1)})$ by $H^{2+}_0(\Omega^{(1)})$. The notation is as in §8. Denote by $K^{(1)}$ and $L^{(1)}$ respectively the Szegö kernel and co-kernel of $\Omega^{(1)}$. For each $u \in H^{2+}(\Omega^{(1)})$ write:

$$(Tu)(z) = \phi'(z)^{\frac{1}{2}}\, u(\phi(z)) \qquad (z \in \Omega^{(2)}).$$

It is easy to check that T is a unitary map of $H^{2+}(\Omega^{(1)})$ onto $H^2(\Omega^{(2)})$, and that T maps $H^2(\Omega^{(1)})$ onto $H^2_0(\Omega^{(2)})$. For each $u \in H^{2+}(\Omega^{(1)})$:

$$(Tu)(\infty) = \lim_{z\to\infty} \{\phi'(z)^{\frac{1}{2}}\, u(\phi(z))\} = \frac{-\mathrm{res}(u)}{\{-\phi'(\infty)\}^{\frac{1}{2}}}. \tag{2}$$

We should make it clear what is meant by $\{-\phi'(\infty)\}^{\frac{1}{2}}$. Near ∞, ϕ has an expansion $\phi(z) = \zeta + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots.$ Hence $\{\phi'(z)\}^{\frac{1}{2}} = \frac{(-a_1)^{\frac{1}{2}}}{z} + \ldots,$ where one of the two square roots of $-a_1$ has been chosen arbitrarily. Now $\phi'(\infty)$, of course, means a_1; and by $\{-\phi'(\infty)\}^{\frac{1}{2}}$ we mean the same square root of $-a_1$ as was chosen before. From (2) it follows that the element of $H^2(\Omega^{(2)})$ representing evaluation at ∞ corresponds under T to the element of $H^{2+}(\Omega^{(1)})$ representing the functional $\frac{-\mathrm{res}}{\{-\phi'(\infty)\}^{\frac{1}{2}}}$. That is:

$$\begin{aligned} K^{(2)+}(z,\infty) &= \frac{-1}{\{-\phi'(\infty)\}^{\frac{1}{2}}}\, \phi'(z)^{\frac{1}{2}} \frac{L^{(1)}(\phi(z),\zeta)}{2\pi\, K^{(1)}(\zeta,\zeta)} \\ &= -\frac{1}{\gamma(E)}\{-\phi'(\infty)\}^{\frac{1}{2}}\, \phi'(z)^{\frac{1}{2}}\, L^{(1)}(\phi(z),\zeta) \end{aligned} \tag{3}$$

since $K^{(1)}(\zeta,\zeta) = \frac{\gamma(E)}{2\pi|\phi'(\infty)|}$. Similarly, for each $u \in H^2(\Omega^{(1)})$, $(Tu)'(\infty) = \{-\phi'(\infty)\}^{\frac{1}{2}}\, u(\zeta)$. So the element of $H^2_0(\Omega^{(2)})$ representing derivation at ∞ corresponds under T to $\{-\phi'(\infty)\}^{\frac{1}{2}}$ times the element of $H^2(\Omega^{(1)})$ representing evaluation at ζ. That is:

$$(4) \qquad \gamma(E)K^{(2)+}(z,\infty)f_E(z) = \overline{\{-\phi'(\infty)\}^{\frac{1}{2}}}\ \phi'(z)^{\frac{1}{2}}\ K^{(1)}(\phi(z),\zeta).$$

Analogous calculations show that if $\Omega^{(1)}$ and $\Omega^{(2)}$ are domains of type G containing ∞, and if ϕ is a conformal map of $\Omega^{(2)}$ onto $\Omega^{(1)}$ taking ∞ to ∞, then the Szegö kernels $K^{(1)+}(z,\infty)$ and $K^{(2)+}(z,\infty)$ of $\Omega^{(1)}$ and $\Omega^{(2)}$ respectively at ∞ are related by the identity:

$$(5) \qquad K^{(2)+}(z,\infty) = \phi'(z)^{\frac{1}{2}}\ \overline{\{\phi'(\infty)^{\frac{1}{2}}\}}\ K^{(1)+}(\phi(z),\infty).$$

The definition of the Szegö kernel at ∞ can be extended to every domain Ω containing ∞ such that $\gamma(S^2 - \Omega) > 0$, in such a way that the Szegö kernels at ∞ of an increasing sequence of domains containing ∞ converge to the Szegö kernel at ∞ of the union Ω, provided that $\gamma(S^2 - \Omega) > 0$. (Apply a linear fractional transformation to the result of Theorem 9.4.) (3), (4) and (5) remain valid: compare Theorem 9.5. The Szegö kernel at ∞ cannot be defined for a domain Ω whose complement E has zero analytic capacity: the problem is the factor $\frac{1}{\gamma(E)}$ in the right side of (3). This problem disappears if we consider the Garabedian function instead; if we define the Garabedian function ψ_E of an arbitrary compact plane set E as $\frac{2\pi}{i}(\gamma(E)K^+(z,\infty))^2$ if $\gamma(E) > 0$, where $K^+(z,\infty)$ is the Szegö kernel at ∞ of $S^2 - E$, and as $\frac{1}{2\pi i}$ if $\gamma(E) = 0$, then the Garabedian functions of a decreasing sequence of compact plane sets converge uniformly to the Garabedian function of the intersection. The proof is again just a matter of rotating the result of Theorem 9.4.

An alternative approach to the study of the kernels at ∞

is to leave the domain fixed and to examine the behaviour of its kernels near ∞. In Proposition 10.1 and Proposition 10.2 below, E is a compact plane set, $\Omega = \Omega(E)$, and K, L, and $K^+(z,\infty)$ are respectively the Szegö kernel, co-kernel and kernel at ∞ of Ω.

10.1 <u>Proposition</u> Near (∞,∞), $K(z,\zeta)$ has an expansion:

$$K(z,\zeta) = \frac{\gamma(E)}{2\pi z\bar{\zeta}} + \text{ third- and higher-order terms.}$$

<u>Proof</u> $K(z,\zeta)$ is analytic in z and $\bar{\zeta}$ and so it can be expanded near (∞,∞) by Laurent's theorem. Since $K(z,\infty) = 0$ for all z and $K(\infty,\zeta) = 0$ for all ζ, the coefficients of 1, $\frac{1}{z}$, $\frac{1}{z^2}$, $\frac{1}{\bar{\zeta}}$ and $\frac{1}{\bar{\zeta}^2}$ in the expansion all vanish. So we need only check the coefficient of $\frac{1}{z\bar{\zeta}}$. For each positive $\zeta \in \Omega$ let f_ζ be the function analytic on Ω and bounded by 1 which maximises the derivative at ζ. An easy "normal families" argument shows that $-f_\zeta \to f_E$ uniformly on compact sets as $\zeta \to \infty$. Hence $2\pi|\zeta|^2K(\zeta,\zeta) = |\zeta|^2|f_\zeta'(\zeta)| \to f_E'(\infty) = \gamma(E)$.

Useful alternative statements of Proposition 10.1 are

$$\gamma(E) = \lim_{\zeta\to\infty} 2\pi|\zeta|^2K(\zeta,\zeta) \quad \text{and} \quad \left.\frac{d}{dz}\right|_{z=\infty}\left.\frac{d}{d\bar{\zeta}}\right|_{\zeta=\infty} K(z,\zeta) = \frac{\gamma(E)}{2\pi}.$$

10.2 <u>Proposition</u> As $\zeta \to \infty$, $\zeta L(z,\zeta) \to -\gamma(E)K^+(z,\infty)$ uniformly on compact subsets of $\Omega(E) - \{\infty\}$, and $\bar{\zeta}K(z,\zeta) \to \gamma(E)K^+(z,\infty)f_E(z)$ uniformly on compact subsets of Ω.

<u>Proof</u> We may assume that $\Omega \in G$. We shall prove the first assertion: the second is similar. We shall in fact prove convergence in $L^2(\partial\Omega)$: this is stronger than is asked.

$$\|\zeta L(z,\zeta)+\gamma(E)K^+(z,\infty)\|_2^2 = \int_{\partial\Omega}(\zeta L(z,\zeta)+\gamma(E)K^+(z,\infty))(\bar{\zeta}\,\overline{L(z,\zeta)}+\gamma(E)\overline{K^+(z,\infty)})\,ds$$

$$= |\zeta|^2\int_{\partial\Omega}|L(z,\zeta)|^2 ds + \bar{\zeta}\gamma(E)\int_{\partial\Omega}\overline{L(z,\zeta)}K^+(z,\infty)ds + \zeta\gamma(E)\int_{\partial\Omega}L(z,\zeta)\overline{K^+(z,\infty)}ds + \gamma(E)^2\int_{\partial\Omega}|K^+(z,\infty)|^2 ds$$

$$= |\zeta|^2 K(\zeta,\zeta) - 2\pi\gamma(E)\bar{\zeta}\frac{d}{dz}\Big|_{z=\infty}(K^+(z,\infty)K(z,\zeta)) - 2\pi\gamma(E)\zeta\overline{\frac{d}{dz}\Big|_{z=\infty}(K^+(z,\infty)K(z,\zeta))} + \gamma(E)^2 K^+(\infty,\infty)$$

(replacing $\overline{L(z,\zeta)}ds$ by $\frac{1}{i}K(z,\zeta)dz$ and using Cauchy's theorem)

$$= |\zeta|^2 K(\zeta,\zeta) - 2\pi\gamma(E)K^+(\infty,\infty)\bar{\zeta}\frac{d}{dz}\Big|_{z=\infty}K(z,\zeta) - 2\pi\gamma(E)K^+(\infty,\infty)\overline{\zeta\frac{d}{dz}\Big|_{z=\infty}K(z,\zeta)} + \gamma(E)^2 K^+(\infty,\infty)$$

$\to \gamma(E)^2K^+(\infty,\infty) - \gamma(E)^2K^+(\infty,\infty) - \gamma(E)^2K^+(\infty,\infty) + \gamma(E)^2K^+(\infty,\infty)$ by Proposition 10.1

$= 0.$

Proposition 10.2, and a comparison of (3) and (4) with Theorem 8.1, show that the analogue of $K(z,\zeta)$ is $\gamma(E)K^+(z,\infty)f_E(z)$ and the analogue of $L(z,\zeta)$ is $-\gamma(E)K^+(z,\infty)$. The customary notation is therefore unfortunate.

If $\infty \in \Omega$ and ζ is a finite point of Ω, then it might seem natural to look at the element representing evaluation at ζ in $H^2(\Omega)$ instead of in $H^2_0(\Omega)$. It is clear that this element, which may be denoted by $K^+(z,\zeta)$, is given by:

$$K^+(z,\zeta) = K(z,\zeta) + \frac{K^+(z,\infty)\overline{K^+(\zeta,\infty)}}{K^+(\infty,\infty)}.$$

This kernel, however, does not emulate the good behaviour of $K(z,\zeta)$ under conformal mapping (Theorem 8.1) and is generally an altogether less natural object.

If $\Omega = S^2 - \bar{D}$ then the domain functions can be computed explicitly. $\{(2\pi)^{-\frac{1}{2}} z^{-n} : n = 0, 1, 2, \ldots\}$ is an orthonormal basis for H^2, and $\{(2\pi)^{-\frac{1}{2}} z^{-n} : n = 1, 2, \ldots\}$ is an orthonormal basis for H^2_0. Hence $K^+(z,\infty) = \frac{1}{2\pi}$ and $K(z,\zeta) = \frac{1}{2\pi}\sum_{n=1}^{\infty} z^{-n}\bar{\zeta}^{-n} = \frac{1}{2\pi(z\bar{\zeta}-1)}$. Since $\overline{L_\zeta}ds = -\frac{1}{i}K_\zeta dz$ on the unit circle, $L(z,\zeta) = 1/(2\pi(z-\zeta))$ when z is on the unit circle. Equality holds for $z \in \Omega$ by analyticity.

CHAPTER III

A PERTURBATION TECHNIQUE

In this chapter we study the effect, on the kernel functions of a domain, of perturbing that domain by removing a small disc from it. We also take a first look at the connection between the resulting theory and the subadditivity problem. Most of the results in this chapter are original: the material of §11 has appeared in my paper [22], and the rest is unpublished. Beware of the notation: the quantity denoted by $K(z,\zeta)$ in [22] is what we are here calling $K^{+}(z,\zeta)$.

§11 <u>The Slope Function</u>

In this section and the next we revert to using ∞ as reference point for technical ease. The purpose of this section is to establish Theorem 11.3, which gives an expression, up to first order in ϵ, for the analytic capacity of a set of the form $E \cup \overline{D}(\eta;\epsilon)$, where E is a compact plane set and $\eta \in \Omega(E)$. We shall give a very detailed proof this time in order to exhibit the technique; in similar proofs afterwards we shall content ourselves with an outline. First we need a lemma which gives bounds on the Szegö kernel function.

11.1 <u>Proposition</u> Let E be a compact plane set, $\zeta \in \Omega(E)$, $\zeta \neq \infty$. Let r and R be the least and greatest distances of points of E from ζ. Then the Szegö kernel K of $\Omega(E)$ satisfies:

(a) $K(\zeta,\zeta) \leqslant \gamma(E)/2\pi r^2$

(b) $K(\zeta,\zeta) \leqslant 1/2\pi r$

(c) $K(\zeta,\zeta) \geqslant \gamma(E)/2\pi R^2$.

Proof (a) $K(\zeta,\zeta) = \frac{1}{2\pi}\sup\{|g'(\zeta)| : g$ is analytic on $\Omega(E)$ and $|g| < 1\} \leqslant \frac{\gamma(E)}{2\pi r^2}$ by Proposition 3.3 (b).

(b) If g is analytic and bounded by 1 on $\Omega(E)$, then in particular g is analytic and bounded by 1 on $D(\zeta;r)$, so that $|g'(\zeta)| \leqslant \frac{1}{r}$ by Schwarz's lemma.

(c) Let E_1 be the inversion of E in the circle $\{z : |z-\zeta| = 1\}$. Applying (a) to the Szegö kernel function K_1 of $\Omega(E_1)$, we have $\gamma(E) = 2\pi K_1(\zeta,\zeta) \leqslant \frac{\gamma(E_1)}{(1/R)^2} = 2\pi R^2 K(\zeta,\zeta)$.

There is also a simple bound for ψ_E: for if $\Omega(E) \in G$ then, in the above notation, $\left|\psi_E(\zeta) - \frac{1}{2\pi i}\right| = \left|\frac{1}{2\pi i}\int_{\partial\Omega(E)}\frac{\psi_E(z)dz}{z-\zeta}\right| \leqslant \frac{1}{2\pi r}\int_{\partial\Omega(E)}|\psi_E|ds = \frac{\gamma(E)}{2\pi r}$. The same bound follows for every compact plane set E. Since $E \subset \overline{D}(\zeta;R)$, $\gamma(E) \leqslant R$ and so we have $|\psi_E(\zeta)| \leqslant \frac{1}{2\pi} + \frac{\gamma(E)}{2\pi r} \leqslant \frac{R}{\pi r}$. Also $K^+(\zeta,\zeta) = K(\zeta,\zeta) + 2\pi\gamma(E)|K^+(\zeta,\infty)|^2$ $= K(\zeta,\zeta) + |\psi_E(\zeta)|/\gamma(E) \leqslant \frac{1}{2\pi r} + \frac{R}{\pi r\gamma(E)} \leqslant \frac{3R}{2\pi r\gamma(E)}$.

We shall also need the following result on Hilbert spaces.

11.2 **Proposition** Let h be a separable Hilbert space, and let $\{u_n\}$ be a sequence of vectors in h whose closed linear span is h. Suppose that the infinite matrix T given by $T_{ij} = (u_j,u_i)$ is bounded and invertible (as an operator on ℓ^2). Let f and g be bounded linear functionals on h. Then the sequences $\{f(u_i)\}$ and $\{g(u_i)\}$ are square-summable and:

$$(f,g) = \sum_{i,j=1}^{\infty}(T^{-1})_{ij}f(u_i)\overline{g(u_j)}.$$

Proof T is positive, since if $\{\alpha_i\}$ is any sequence of complex numbers which is eventually zero then $\sum T_{ij}\overline{\alpha_i}\alpha_j = \left(\sum\alpha_j u_j, \sum\alpha_i u_i\right) \geqslant 0$ and such sequences constitute a dense subset of ℓ^2. T is the matrix of some positive invertible bounded linear operator P on ℓ^2. P has a positive square root $P^{\frac{1}{2}}$, which is invertible since P is invertible. For $i = 1, 2, 3, \ldots$, write $w_i = P^{\frac{1}{2}} e_i$, where e_i is the vector with 1 in its ith place and 0 elsewhere. Since $P^{\frac{1}{2}}$ is invertible, ℓ^2 is the closed linear span of $\{w_i\}$. $(w_j, w_i) = (P^{\frac{1}{2}} e_j, P^{\frac{1}{2}} e_i) = (Pe_j, e_i) = T_{ij} = (u_j, u_i)$; so we can define a unitary J from ℓ^2 onto h by $J(w_i) = u_i$ for all i, extended to the whole of ℓ^2 by linearity and continuity. The bounded linear functionals J^*f and J^*g on ℓ^2 are represented by some elements s and t respectively of ℓ^2. $(e_i, P^{\frac{1}{2}} s) = (P^{\frac{1}{2}} e_i, s) = (w_i, s) = (J^*f)(w_i) = f(Jw_i) = f(u_i)$. Similarly $(e_i, P^{\frac{1}{2}} t) = g(u_i)$. Hence $\{f(u_i)\}$ and $\{g(u_i)\}$ are square-summable. Also:

$$\begin{aligned}
(f,g) &= (t,s) \\
&= (P^{-1}(P^{\frac{1}{2}} t), P^{\frac{1}{2}} s) \\
&= \sum_{i,j=1}^{\infty} (T^{-1})_{ij} (e_i, P^{\frac{1}{2}} s) \overline{(e_j, P^{\frac{1}{2}} t)} \\
&= \sum_{i,j=1}^{\infty} (T^{-1})_{ij} f(u_i) \overline{g(u_j)}.
\end{aligned}$$

11.3 <u>Theorem</u> Let E be a compact plane set. Then there is a positive real-valued function $a_E(\eta)$, the <u>slope function</u> of E, defined on $\Omega(E) - \{\infty\}$, with the property that for all $\eta \in \Omega(E) - \{\infty\}$:

$$\gamma(E \cup \overline{D}(\eta;\epsilon)) = \gamma(E) + \epsilon a_E(\eta) + O(\epsilon^2).$$

$a_E(\eta)$ is given explicitly by:

$$a_E(\eta) = 2\pi |\psi_E(\eta)| \{1 - |f_E(\eta)|^2\}.$$

The bounds for the error term depend only on $\gamma(E)$ and on the ratio between the greatest and least distances of points of E from η.

<u>Proof</u> If $\gamma(E) = 0$ then the theorem asserts that

$\gamma(E \cup \overline{D}(\eta;\epsilon)) = \epsilon + O(\epsilon^2)$, which holds trivially since $\gamma(E \cup \overline{D}(\eta;\epsilon)) = \epsilon$ by Corollary 2.11. So we can assume that $\gamma(E) > 0$. We may suppose that $\eta = 0$. Let r and R be respectively the least and greatest distances of points of E from 0. We shall prove the theorem by showing that:

$$\epsilon \leqslant \frac{1}{300}(r/R)^2\gamma(E) \Rightarrow |\gamma(E\cup\overline{D}(0;\epsilon)) - \gamma(E) - \epsilon a_E(0)| \leqslant 10^5(R/r)^4\gamma(E)^{-1}\epsilon^2.$$

We may suppose that $\Omega(E) \subset G$: the convergence of the Garabedian function and the Ahlfors function of a decreasing sequence of compact sets then guarantees the general result.

Fix $\epsilon \leqslant \frac{1}{300}(r/R)^2\gamma(E)$. Since $r \leqslant R$ and $\gamma(E) \leqslant R$, we have $\epsilon \leqslant r/300$; so $\overline{D}(0;\epsilon)$ does not meet E. Write $E_1 = E\cup\overline{D}(0;\epsilon)$, $\Omega = \Omega(E)$, $\Omega_1 = \Omega(E_1)$, $H^2 = H^2(\Omega)$, $H_1^2 = H^2(\Omega_1)$, $\gamma = \gamma(E)$, and $\gamma_1 = \gamma(E_1)$. Choose an orthonormal basis $\{u_n\}_{n=1}^{\infty}$ for H^2. Now we can use the Cauchy integral to express any element of H_1^2 as the sum of an element of H^2 and an element of $H^2(S^2 - \overline{D}(0;\epsilon))$. The latter space is the closed linear span of $\{z^{-n} : n \geqslant 0\}$. It follows that if, for $n \geqslant 1$, v_n is any function analytic on $\overline{\Omega}$ except for a pole of order n at 0, then H_1^2 is the closed linear span of $\{u_n\} \cup \{v_n\}$. To be specific, we shall put:

$$v_n(z) = \frac{\epsilon^{n-\frac{1}{2}}}{\sqrt{2\pi}\, K(0,0)} \frac{K(z,0)}{z^n}.$$

Now $\frac{1}{2\pi\gamma_1}$ is the square of the norm of evaluation at ∞ in H_1^2. Our proof consists of calculating this by applying Proposition 11.2 to $\{u_n\} \cup \{v_n\}$.

We shall calculate various bounds now, so as not to break continuity later. Throughout, "$\| \; \|$" and "norm" will refer to the norm of an element of a Hilbert space, or the norm of an infinite matrix considered as a bounded operator on ℓ^2; and "$\| \|_\infty$" will denote the supremum of the absolute value of a function on the set $\bar{D}(0;\epsilon)$.

Let $z_0 \in \underset{\sim}{C}$, $|z_0| \leq \epsilon$. For $n \geq 1$:

$$u_n(z_0) = \frac{1}{2\pi i}\int_{|z|=\frac{r}{2}} \frac{u_n(z)dz}{z - z_0}.$$

Hence:

$$\|u_n\|_\infty \leq \frac{1}{2\pi(r/2 - \epsilon)}\int_{|z|=\frac{r}{2}} |u_n| ds,$$

so that by Schwarz's inequality:

$$\|u_n\|_\infty^2 \leq \frac{2\pi r/2}{4\pi^2(r/2 - \epsilon)^2}\int_{|z|=\frac{r}{2}} |u_n|^2 ds.$$

Summing over n and using the remarks after Proposition 11.1 gives:

$$\sum \|u_n\|_\infty^2 \leq \frac{r/2}{2\pi(r/2 - \epsilon)^2}\int_{|z|=\frac{r}{2}} K^+(z,z)ds$$

$$\leq \frac{(r/2)^2}{(r/2 - \epsilon)^2}\,\frac{3(R + r/2)}{2\pi(r - r/2)\gamma}$$

(1) $$\leq \frac{10\,R}{2\pi r\gamma}$$

since $\epsilon \leq r/300$ and $r \leq R$. Analogous computation gives:

(2) $$\sum \|u_n{}'\|_\infty^2 \leq \frac{40R}{2\pi r^3\gamma}.$$

A fortiori:

(3) $$\sum |u_n'(0)|^2 \leq \frac{40R}{2\pi r^3\gamma}.$$

Next we want a bound for $\left\|\frac{d^k}{dz^k} K(z,0)\right\|_\infty$. Let $z_0 \in \underset{\sim}{C}$, $|z_0| \leq \epsilon$. Then for $k \geq 1$ and for all $s < r$:

$$\left|\frac{d^k}{dz^k} K(z,0)\bigg|_{z=z_0}\right| = \left|\frac{k!}{2\pi i}\int_{|z|=s}\frac{K(z,0)dz}{(z - z_0)^{k+1}}\right| \leq \frac{k!}{2\pi}\,\frac{2\pi s}{(s-\epsilon)^{k+1}}\,\frac{1}{2\pi(r-s)}.$$

(Here we have estimated $|K(z,0)|$ by $K(z,z)^{\frac{1}{2}} K(0,0)^{\frac{1}{2}}$ and then used Proposition 11.1 (b).) In particular, putting $s = \dfrac{kr}{k+1}$, we have:

$$\left|\frac{d^k}{dz^k} K(z,0)\Big|_{z=z_0}\right| \leq \frac{k!}{2\pi} \frac{1}{(r-(k+1)\epsilon/k)^{k+1}} \frac{(k+1)^{k+1}}{k^k}$$
$$\leq \frac{e(k+1)!}{2\pi} \frac{1}{(r-2\epsilon)^{k+1}}$$
$$\leq \frac{3(k+1)!}{2\pi(r-2\epsilon)^{k+1}}.$$

This holds also for $k = 0$ by Proposition 11.1 (b). Hence for $k \geq 0$:

$$\left\|\frac{d^k}{dz^k} K(z,0)\right\|_\infty \leq \frac{3(k+1)!}{2\pi(r-2\epsilon)^{k+1}}. \tag{4}$$

We need one more estimate. Since $\epsilon < r/300$, (4) gives:

$$\left\|\frac{dK(z,0)}{dz}\right\|_\infty \leq \frac{7}{2\pi r^2}.$$

So by Proposition 11.1 (c):

$$\|K(z,0)\|_\infty \leq K(0,0) + \epsilon\left\|\frac{dK(z,0)}{dz}\right\|_\infty$$
$$\leq K(0,0)\left(1 + \epsilon \frac{7}{2\pi r^2} \frac{2\pi R^2}{\gamma}\right)$$
$$\leq \frac{11}{10} K(0,0). \tag{5}$$

We shall imagine the basis $\{u_n\} \cup \{v_n\}$ to be partitioned into three sections. The first section consists of all the u_n, the second section consists of v_1 alone, and the third section consists of the rest of the v_n. The corresponding matrix T of inner products will be in block form:

$$T = I + M, \qquad M = \begin{bmatrix} A & B^* & C^* \\ B & D & E^* \\ C & E & F \end{bmatrix}. \tag{6}$$

Next we calculate the inner products. Denote inner products in H_1^2 by $(\, , \,)$. By a statement of the form "X = Y with error Z" we shall mean $|X - Y| \leq Z$, or $\|X - Y\| \leq Z$, according to context.

$$(u_n, u_m) = \int_{\partial\Omega_1} u_n \overline{u_m} \, ds$$

$$= \int_{\partial\Omega} u_n \overline{u_m} \, ds + \int_{|z|=\epsilon} u_n \overline{u_m} \, ds$$

$$= \delta_{mn} + 2\pi\epsilon \overline{u_m(0)} u_n(0)$$
$$+ \epsilon \int_0^{2\pi} [\overline{u_m(0)}(u_n(\epsilon e^{i\theta}) - u_n(0)) + (\overline{u_m(\epsilon e^{i\theta})} - \overline{u_m(0)}) u_n(\epsilon e^{i\theta})] \, d\theta .$$

$$|(u_n, u_m) - \delta_{mn} - 2\pi\epsilon \overline{u_m(0)} u_n(0)| \leq 2\pi\epsilon^2 (\|u_m\|_\infty \|u_n'\|_\infty + \|u_m'\|_\infty \|u_n\|_\infty).$$

Now the norm of the matrix $[2\pi\epsilon \overline{u_m(0)} u_n(0)]$ is $2\pi\epsilon \left(\sum |u_m(0)|^2\right)^{\frac{1}{2}} \left(\sum |u_n(0)|^2\right)^{\frac{1}{2}} = 2\pi\epsilon K^+(0,0) \leq \dfrac{3R\epsilon}{r\gamma}$ by the remarks following Proposition 11.1. The norm of the matrix $[2\pi\epsilon^2 (\|u_m\|_\infty \|u_n'\|_\infty + \|u_m'\|_\infty \|u_n\|_\infty)]$ is at most $4\pi\epsilon^2 \left(\sum \|u_m\|_\infty^2\right)^{\frac{1}{2}} \left(\sum \|u_n'\|_\infty^2\right)^{\frac{1}{2}}$ $\leq 40 \dfrac{R\epsilon^2}{r^2\gamma}$ by (1) and (2). So (see the format (6)):

$$(7) \qquad A = [2\pi\epsilon \overline{u_m(0)} u_n(0)] \text{ with error } 40 \, R r^{-2} \gamma^{-1} \epsilon^2 .$$

Also, $\|A\| \leq \dfrac{3R\epsilon}{r\gamma} + 40 \dfrac{R\epsilon^2}{r^2\gamma} \leq \dfrac{4R\epsilon}{r\gamma} \leq \dfrac{4R^2\epsilon}{r^2\gamma}$ since $\epsilon < r/300$ and $r < R$. In fact the cruder bound $\|A\| \leq 30(R/r)^2 \gamma^{-1} \epsilon$ will be sufficient. Observe that, since $\epsilon \leq \dfrac{1}{300}(r/R)^2 \gamma$, we have also $\|A\| \leq \dfrac{1}{10}$.

The nth element of B is:

$$(u_n, v_1) = \frac{\epsilon^{\frac{1}{2}}}{\sqrt{2\pi}\, K(0,0)} \int_{\partial\Omega} \frac{\overline{K(z,0)} u_n(z) \, ds}{\overline{z}} + \frac{\epsilon^{\frac{1}{2}}}{\sqrt{2\pi}\, K(0,0)\epsilon i} \int_{|z|=\epsilon} \overline{K(z,0)} u_n(z) \, dz .$$

Now the second term on the right hand side is:

$$\frac{\epsilon^{-\frac{1}{2}}}{\sqrt{2\pi}\, K(0,0) i} \int_{|z|=\epsilon} (\overline{K(z,0)} - \overline{K(0,0)}) u_n(z) \, dz$$

by Cauchy's theorem, and is therefore bounded in magnitude by

$\frac{2\pi\epsilon^{\frac{3}{2}}}{\sqrt{2\pi}\ K(0,0)}\ \frac{7}{2\pi r^2}\|u_n\|_\infty$ by the remark after (4). So we have:

$$B = \left[\frac{\epsilon^{\frac{1}{2}}}{\sqrt{2\pi}\ K(0,0)}\int_{\partial\Omega}\frac{\overline{K(z,0)u_n(z)ds}}{\bar{z}}\right] \text{ with error } \frac{2\pi\epsilon^{\frac{3}{2}}}{\sqrt{2\pi}\ K(0,0)}\ \frac{7}{2\pi r^2}\left(\sum\|u_n\|_\infty^2\right)^{\frac{1}{2}}$$

$$\leq 7\sqrt{10}\ (R/r)^{\frac{5}{2}}\ \gamma^{-\frac{3}{2}}\epsilon^{\frac{3}{2}}$$

(8) $$\leq 23(R/r)^3\ \gamma^{-\frac{3}{2}}\epsilon^{\frac{3}{2}}$$

using Proposition 11.1 (c), (1), and the fact that $r < R$. The norm of the matrix in square brackets is:

$$\frac{\epsilon^{\frac{1}{2}}}{\sqrt{2\pi}\ K(0,0)}\left(\int_{\partial\Omega}\frac{|K(z,0)|^2 ds}{|z|^2}\right)^{\frac{1}{2}}$$

$$\leq \frac{\epsilon^{\frac{1}{2}}}{\sqrt{2\pi}\ K(0,0)r}\left(\int_{\partial\Omega}|K(z,0)|^2 ds\right)^{\frac{1}{2}}$$

$$= \frac{\epsilon^{\frac{1}{2}}}{\sqrt{2\pi}\ K(0,0)^{\frac{1}{2}}\ r}$$

$$\leq (R/r)\gamma^{-\frac{1}{2}}\epsilon^{\frac{1}{2}}$$

by Proposition 11.1 (c). Hence $\|B\| \leq (R/r)\gamma^{-\frac{1}{2}}\epsilon^{\frac{1}{2}} + 23(R/r)^3\gamma^{-\frac{3}{2}}\epsilon^{\frac{3}{2}}$ $\leq \frac{11}{10}(R/r)\gamma^{-\frac{1}{2}}\epsilon^{\frac{1}{2}}$ since $\epsilon \leq \frac{1}{300}(r/R)^2\gamma$. The cruder bounds $\|B\| \leq \frac{1}{10}$ and $\|B\|^2 \leq 30(R/r)^2\gamma^{-1}\epsilon$ will suffice. Also, using (8) and the estimates calculated in the last few lines, we have:

$$B^*B = \left[\frac{\epsilon}{2\pi K(0,0)^2}\int_{\partial\Omega}\frac{K(z,0)\overline{u_m}ds}{z}\int_{\partial\Omega}\frac{\overline{K(z,0)u_n ds}}{\bar{z}}\right]$$

$$\text{with error } 2.(R/r)\gamma^{-\frac{1}{2}}\epsilon^{\frac{1}{2}}.23(R/r)^3\gamma^{-\frac{3}{2}}\epsilon^{\frac{3}{2}} + (23(R/r)^3\gamma^{-\frac{3}{2}}\epsilon^{\frac{3}{2}})^2$$

(9) $$\leq 50(R/r)^4\gamma^{-2}\epsilon^2.$$

The elements of C are, for $n \geq 1$ and $m \geq 2$:

$$(u_n, v_m) = \frac{\epsilon^{m-\frac{1}{2}}}{\sqrt{2\pi}\ K(0,0)}\int_{\partial\Omega}\frac{\overline{K(z,0)u_n(z)ds}}{\bar{z}^m} + \frac{\epsilon^{-m+\frac{1}{2}}}{\sqrt{2\pi}\ K(0,0)i}\int_{|z|=\epsilon}\overline{K(z,0)}u_n(z)z^{m-1}dz.$$

Call the first and second terms of the above expression P_{mn} and Q_{mn} respectively. Then:

$$\|P\| \leqslant \frac{1}{\sqrt{2\pi}\,K(0,0)}\left(\sum_{m=2}^{\infty}\int_{\partial\Omega}\frac{|K(z,0)|^2 ds}{|z|^{2m}}\,\epsilon^{2m-1}\right)^{\frac{1}{2}}$$
$$\leqslant \frac{1}{\sqrt{2\pi}\,K(0,0)}\left(\sum_{m=2}^{\infty}\frac{\epsilon^{2m-1}}{r^{2m}}\,K(0,0)\right)^{\frac{1}{2}}$$
$$= \frac{1}{\sqrt{2\pi}\,K(0,0)^{\frac{1}{2}}}\,\frac{\epsilon^{\frac{3}{2}}}{r(r^2-\epsilon^2)^{\frac{1}{2}}}$$
$$\leqslant \frac{2R}{\gamma^{\frac{1}{2}}}\,\frac{\epsilon^{\frac{3}{2}}}{r^2}$$
$$\leqslant (R/r)^2\gamma^{-1}\epsilon.$$

We estimate the integral in the expression for Q_{mn} as follows. Replace $K(z,0)$ by $K(z,0)$ minus its Taylor expansion about 0 as far as the term in z^{m-1}. By Cauchy's theorem, the added terms do not affect the integral. By Taylor's theorem, $K(z,0)$ minus its Taylor expansion is bounded on $|z| = \epsilon$ by $\frac{\epsilon^m}{m!}\left\|\frac{d^m}{dz^m}K(z,0)\right\|_\infty$, which by (4) is at most $\frac{3\epsilon^m(m+1)}{2\pi(r-2\epsilon)^{m+1}}$. Hence:

$$\|Q\| \leqslant \frac{1}{\sqrt{2\pi}\,K(0,0)}\,2\pi\epsilon\left(\sum_{m=2}^{\infty}\frac{9\epsilon^{2m-1}(m+1)^2}{4\pi^2(r-2\epsilon)^{2m+2}}\right)^{\frac{1}{2}}\left(\sum\|u_n\|_\infty^2\right)^{\frac{1}{2}}$$
$$\leqslant \frac{\sqrt{2\pi}\,R^2}{\gamma}\,2\pi\epsilon\,\frac{10\epsilon^{\frac{3}{2}}}{2\pi r^3}\left(\frac{10\,R}{2\pi r\gamma}\right)^{\frac{1}{2}}$$
$$\leqslant (R/r)^2\gamma^{-1}\epsilon.$$

Hence $\|C\| \leqslant \|P\| + \|Q\| \leqslant 2(R/r)^2\gamma^{-1}\epsilon$. Once again we shall need only $\|C\| \leqslant 30(R/r)^2\gamma^{-1}\epsilon \leqslant \frac{1}{10}$.

It is convenient to deal with $\begin{bmatrix} D & E^* \\ E & F \end{bmatrix}$ as a single matrix. Its (m,n) th element (see (6)) is, for $m \geqslant 1$, $n \geqslant 1$:

$$\frac{\epsilon^{m+n-1}}{2\pi K(0,0)^2}\int_{\partial\Omega}\frac{|K(z,0)|^2 ds}{\bar{z}^m z^n} + \left(\frac{\epsilon^{m+n-1}}{2\pi K(0,0)^2}\int_{|z|=\epsilon}\frac{|K(z,0)|^2 ds}{\bar{z}^m z^n} - \delta_{mn}\right).$$

Denote by G_{mn} and H_{mn} respectively the first term and the bracketed term of the above expression. We have:

$$|G_{mn}| \leqslant \frac{\epsilon^{m+n-1}}{2\pi K(0,0)^2}\,\frac{1}{r^{m+n}}\int_{\partial\Omega}|K(z,0)|^2 ds = \frac{\epsilon^{m+n-1}}{2\pi K(0,0)r^{m+n}}\,.$$

Hence $\|G\| \leqslant \dfrac{1}{2\pi\epsilon K(0,0)}\displaystyle\sum_{n=1}^{\infty}\frac{\epsilon^{2n}}{r^{2n}} \leqslant \frac{2\epsilon R^2}{r^2\gamma}$. H is trickier to deal with. We have:

$$\begin{aligned} H_{nn} &= \frac{1}{2\pi K(0,0)^2\epsilon}\int_{|z|=\epsilon}|K(z,0)|^2 ds \; - 1 \\ &= \frac{1}{2\pi i K(0,0)^2}\int_{|z|=\epsilon} K(z,0)(\overline{K(z,0)} - \overline{K(0,0)})z^{-1}dz. \end{aligned}$$

So $|H_{nn}| \leqslant \dfrac{1}{2\pi K(0,0)}\; 2\pi\epsilon\;\dfrac{\|K(z,0)\|_\infty}{K(0,0)}\;\epsilon\left\|\dfrac{dK(z,0)}{dz}\right\|_\infty \dfrac{1}{\epsilon} \leqslant 8(R/r)^2\gamma^{-1}\epsilon$ by Proposition 11.1 (c), (5), and (4) with $k = 1$. If $m > n$, then:

$$H_{mn} = \frac{\epsilon^{n-m}}{2\pi i K(0,0)^2}\int_{|z|=\epsilon} K(z,0)\overline{K(z,0)}z^{m-n-1}dz.$$

As before, we may replace the second occurrence of $K(z,0)$ in the integral by $K(z,0)$ minus its Taylor expansion, this time as far as the term in z^{m-n-1}. Then by (5), Proposition 11.1 (c), and (4) with $k = m - n$:

$$\begin{aligned} |H_{mn}| &\leqslant \epsilon^{n-m}\,\frac{R^2}{\gamma}\,\frac{11}{10}\,2\pi\epsilon\,\frac{3\epsilon^{m-n}(m-n+1)}{2\pi(r-2\epsilon)^{m-n+1}}\,\epsilon^{m-n-1} \\ &\leqslant 4(R/r)^2\gamma^{-1}\epsilon(|m-n| + 1)(1/298)^{|m-n|-1} \end{aligned}$$

since $\epsilon \leqslant r/300$. Since H is hermitian, this holds also for $m<n$. Combining the cases $m = n$, $m > n$ and $m < n$, we see that:

$$\begin{aligned} \|H\| &\leqslant (R/r)^2\gamma^{-1}\epsilon\left(8 + 2\times 4\left(2 + \frac{3}{298} + \frac{4}{(298)^2} + \cdots\cdots\right)\right) \\ &\leqslant 25(R/r)^2\gamma^{-1}\epsilon. \end{aligned}$$

So the norm of $\begin{bmatrix} D & E^* \\ E & F \end{bmatrix}$ is at most $\|G\| + \|H\| \leqslant 30(R/r)^2\gamma^{-1}\epsilon$. Hence each of $\|D\|$, $\|E\|$, $\|F\| \leqslant 30(R/r)^2\gamma^{-1}\epsilon \leqslant \frac{1}{10}$.

To summarise: we have shown that:

$$(10)\qquad \begin{aligned}&\|A\|,\ \|B\|^2,\ \|C\|,\ \|D\|,\ \|E\|,\ \|F\| \leqslant 30(R/r)^2\gamma^{-1}\epsilon;\\ &\|A\|,\ \|B\|,\ \|C\|,\ \|D\|,\ \|E\|,\ \|F\| \leqslant \frac{1}{10}.\end{aligned}$$

In particular we have verified that M is a bounded matrix: indeed that $\|M\| \leqslant \frac{3}{10} < 1$. Thus $T = I + M$ is invertible, and Proposition 11.2 applies.

Our next step is to calculate the top left-hand block S of the inverse of T. Since $T^{-1} = I - M + M^2 - M^3 + \ldots\ldots$:

$$\begin{aligned} S \;=\;& I\\ &- A\\ &+ A^2 + B^*B + C^*C\\ &- A^3 - AB^*B - AC^*C - B^*BA - B^*DB - B^*E^*C - C^*CA - C^*EB - C^*FC\\ &+ \ldots\ldots \end{aligned}$$

The row of this expression containing products of degree n $(n \geqslant 4)$ consists of 3^{n-1} terms. The norm of each of these terms is at most $(30)^2(R/r)^4\gamma^{-2}\epsilon^2(1/10)^{n-4}$ by (10). Hence $S = I - A + B^*B$ with error:

$$\frac{(30)^2\epsilon^2R^4}{\gamma^2r^4}\left(1 + 1 + \frac{1}{10} + 1 + \frac{1}{10} + 1 + 1 + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + 27\left(1 + \frac{3}{10} + \left(\frac{3}{10}\right)^2 + \ldots\ldots\right)\right)$$

$\leqslant 39800(R/r)^4\gamma^{-2}\epsilon^2.$

Using (7) and (9), we have:

$$S = \left[\delta_{mn} - 2\pi\epsilon\overline{u_m(0)}u_n(0) + \frac{\epsilon}{2\pi K(0,0)^2}\int_{\partial\Omega}\frac{\overline{K(z,0)\overline{u_m}ds}}{z}\int_{\partial\Omega}\frac{\overline{K(z,0)u_nds}}{\overline{z}}\right]$$

with error $40Rr^{-2}\gamma^{-1}\epsilon^2 + 50(R/r)^4\gamma^{-2}\epsilon^2 + 39800(R/r)^4\gamma^{-2}\epsilon^2$

$\leqslant 40000(R/r)^4\gamma^{-2}\epsilon^2.$

Finally we apply Proposition 11.2, which says that $\frac{1}{2\pi\gamma_1} = \sum_{m,n} S_{mn}u_m(\infty)\overline{u_n(\infty)}$ (since $v_n(\infty) = 0$ for all n). Hence:

$$\frac{1}{2\pi\gamma_1} = \sum|u_m(\infty)|^2 - 2\pi\epsilon\left|\sum\overline{u_m(0)}u_m(\infty)\right|^2 + \frac{\epsilon}{2\pi K(0,0)^2}\left|\sum\left(u_m(\infty)\int_{\partial\Omega}\frac{K(z,0)\overline{u_m}ds}{z}\right)\right|^2$$

with error $40000(R/r)^4\gamma^{-2}\epsilon^2\sum|u_m(\infty)|^2 = 40000(R/r)^4\gamma^{-2}\epsilon^2/(2\pi\gamma)$.

Multiplying by $2\pi\gamma$ and using the fact that $\sum u_m(z)\overline{u_m(\zeta)} = K^+(z,\zeta)$, we have:

$$(11)\qquad \frac{\gamma}{\gamma_1} = 1 - 4\pi^2\gamma\epsilon|K^+(0,\infty)|^2 + \frac{\gamma\epsilon}{K(0,0)^2}\left|\int_{\partial\Omega}\frac{K(z,0)\overline{K^+(z,\infty)}ds}{z}\right|^2$$

with error $40000(R/r)^4\gamma^{-2}\epsilon^2$.

Now the last term simplifies. On $\partial\Omega$, $f_E(z)\psi_E(z)dz = 0$, so that $ds = \frac{\psi_E(z)}{|\psi_E(z)|}f_E(z)dz = \frac{K^+(z,\infty)}{\overline{K^+(z,\infty)}}\frac{f_E(z)}{i}dz$. Therefore:

$$\int_{\partial\Omega}\frac{K(z,0)\overline{K^+(z,\infty)}ds}{z} = \frac{1}{i}\int_{\partial\Omega}\frac{K(z,0)K^+(z,\infty)f_E(z)dz}{z} = -2\pi K(0,0)K^+(0,\infty)f_E(0)$$

since $K(z,0)K^+(z,\infty)f_E(z)$ is analytic on $\overline{\Omega}$ and vanishes at ∞. Substituting in (11), we have:

$$\frac{\gamma}{\gamma_1} = 1 - 4\pi^2\gamma\epsilon|K^+(0,\infty)|^2\{1 - |f_E(0)|^2\} \quad \text{with error } 40000(R/r)^4\gamma^{-2}\epsilon^2$$
$$= 1 - 2\pi\gamma^{-1}\epsilon|\psi_E(0)|\{1 - |f_E(0)|^2\} \quad \text{with error } 40000(R/r)^4\gamma^{-2}\epsilon^2.$$

Now $2\pi\gamma^{-1}\epsilon|\psi_E(0)|\{1 - |f_E(0)|^2\} \leq 2\pi\gamma^{-1}\epsilon|\psi_E(0)| \leq 2(R/r)\gamma^{-1}\epsilon \leq \frac{1}{150}$. Also $40000(R/r)^4\gamma^{-2}\epsilon^2 \leq \frac{1}{2}$. So we can invert to obtain, by elementary arithmetic:

$$\left(\frac{\gamma}{\gamma_1}\right)^{-1} = 1 + 2\pi\gamma^{-1}\epsilon|\psi_E(0)|\{1 - |f_E(0)|^2\} \quad \text{with error } 10^5(R/r)^4\gamma^{-2}\epsilon^2$$
$$\gamma_1 = \gamma + 2\pi\epsilon|\psi_E(0)|\{1 - |f_E(0)|^2\} \quad \text{with error } 10^5(R/r)^4\gamma^{-1}\epsilon^2,$$

as required.

It is as well to explain the curious choice of the functions v_n in the above proof. The only essential property of v_n that we used is that it vanishes at ∞ and is analytic on $\overline{\Omega}$

except for a pole at 0 near which $v_n(z) = 1/\sqrt{2\pi}\ \epsilon^{n-\frac{1}{2}} z^{-n} + \ldots\ldots$ However, the simpler choice $v_n(z) = 1/\sqrt{2\pi}\ \epsilon^{n-\frac{1}{2}} z^{-n}$ would have yielded an error bound dependent on the length of $\partial\Omega$. The proof would then have been valid only in the case when $\Omega(E) \in G$: the passage to the general case would not have been possible as the error bound would not be uniform.

In the proof of Theorem 11.3 it is not in fact necessary to assume the convergence of the Garabedian function. From the result of Theorem 11.3 in the case when $\Omega(E) \in G$ there follow easily both the general case of Theorem 11.3 <u>and</u> the convergence of the Garabedian function. This was the approach used in my paper [22], which was written before Suita's proof of the convergence of the Garabedian function had appeared.

§12 <u>Perturbation of the Szegö Kernel and Co-kernel</u>

This section extends the result of Theorem 11.3. The techniques involved are precisely the same as before, and so we shall give only outlines of the proofs.

Up till now we have talked of the Szegö kernel and co-kernel of a <u>domain</u>. We are now going to bring the notation into line with our notations for the Ahlfors function and the Garabedian function. Let E be a compact subset of S^2, and let $z, \zeta \in S^2 - E$. We define $K_E(z,\zeta)$ as follows. If z and ζ lie in the same component of $S^2 - E$, then $K_E(z,\zeta) = K(z,\zeta)$, where K is the Szegö kernel of that component. If z and ζ lie in distinct components of $S^2 - E$ then $K_E(z,\zeta) = 0$. Suppose in addition that $z \neq \zeta$. Then we define $L_E(z,\zeta)$ as follows. If z and ζ lie in the same component of $S^2 - E$, then $L_E(z,\zeta) = L(z,\zeta)$, where L is the co-kernel of that component. If z and ζ lie in distinct components of $S^2 - E$ then $L_E(z,\zeta) = 0$. Finally suppose that $\infty \notin E$ and that $\gamma(E) > 0$. If $z \in \Omega(E)$ then we define $K_E^+(z,\infty) = K^+(z,\infty)$, where K^+ is the Szegö kernel of $\Omega(E)$ at ∞. If z is in a bounded component of $S^2 - E$ then we define $K_E^+(z,\infty) = 0$.

We now introduce another piece of notation. Let f be any complex-valued set function defined on some class K of compact subsets of S^2. Let $E \in K$, and let $\eta \in \underset{\sim}{C} - E$. Suppose that for all sufficiently small $\epsilon > 0$ $E \cup \overline{D}(\eta;\epsilon) \in K$, and that $f(E \cup \overline{D}(\eta;\epsilon)) = f(E) + \alpha\epsilon + O(\epsilon^2)$ for some $\alpha \in \underset{\sim}{C}$. Then we define the <u>perturbation</u> of $f(E)$ at η to be:

$$\mathrm{Pert}(f(E);\eta) = \alpha.$$

(The notation "$\mathrm{Pert}(f;E;\eta)$" would be formally more correct, but would prove too cumbersome in practice.) In this notation, Theorem 11.3 says that $\mathrm{Pert}(\gamma(E);\eta) = a_E(\eta) = 2\pi|\psi_E(\eta)|\{1 - |f_E(\eta)|^2\}$.

12.1 <u>Theorem</u> Let E be any compact subset of S^2. Let $\xi, \zeta \in S^2 - E$. Let $\eta \in \underset{\sim}{C} - E$, $\eta \neq \xi$, $\eta \neq \zeta$. Then:

$$\mathrm{Pert}(K_E(\xi,\zeta);\eta) = 2\pi(L_E(\eta,\xi)\overline{L_E(\eta;\zeta)} - \overline{K_E(\eta,\xi)}K_E(\eta,\zeta)).$$

<u>Proof</u> We can assume that ξ, ζ and η all lie in the same component Ω of $S^2 - E$, as otherwise both sides are zero. As in §11 we can assume that $\gamma(E) > 0$. Write $\Omega_1 = \Omega - \overline{D}(\eta;\epsilon)$. Denote the Szegö kernels of Ω and Ω_1 by K and K_1 respectively and the corresponding co-kernels by L and L_1. We claim that $K_1(\xi,\zeta) = K(\xi,\zeta) + 2\pi\epsilon(L(\eta,\xi)\overline{L(\eta,\zeta)} - \overline{K(\eta,\xi)}K(\eta,\zeta)) + O(\epsilon^2)$ and that the bounds on the error term depend only on $K(\eta,\eta)$, $|\xi - \eta|$, $|\zeta - \eta|$, and the greatest and least distances of each of ξ, ζ and η from

points of E. We shall refer to these as "allowed quantities". We may suppose that $\eta = 0$. As in §11 we may assume that $\Omega \in G$. We shall assume that Ω is bounded: if $\infty \in \Omega$ then the proof is almost identical. Write $H^2 = H^2(\Omega)$, and $H_1^2 = H^2(\Omega_1)$.

Choose an orthonormal basis $\{u_n\}$ for H^2. Let α be a function, analytic on $\overline{\Omega}$, satisfying $\alpha(0) = 1$. For $n = 1, 2, 3, \ldots$ write:

$$v_n(z) = \frac{\epsilon^{n-\frac{1}{2}}}{\sqrt{2\pi}} \frac{\alpha(z)}{z^n}.$$

Partition the basis $\{u_n\} \cup \{v_n\}$ of H_1^2 into three sections, $\{u_n\}$, $\{v_1\}$, and $\{v_n : n \geq 2\}$ as before. The matrix of inner products is:

$$(1) \qquad T = I + \left[\begin{array}{c|c|c} 2\pi\epsilon\overline{u_m(0)}u_n(0) & \dfrac{\epsilon^{\frac{1}{2}}}{\sqrt{2\pi}}\displaystyle\int_{\partial\Omega}\frac{\overline{u_m(z)}\,\alpha(z)ds}{z} & 0 \\ \hline \dfrac{\epsilon^{\frac{1}{2}}}{\sqrt{2\pi}}\displaystyle\int_{\partial\Omega}\frac{u_n(z)\overline{\alpha(z)}ds}{\overline{z}} & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] + \begin{bmatrix} O(\epsilon^2) & O(\epsilon^{\frac{3}{2}}) & O(\epsilon) \\ O(\epsilon^{\frac{3}{2}}) & O(\epsilon) & O(\epsilon) \\ O(\epsilon) & O(\epsilon) & O(\epsilon) \end{bmatrix}$$

exactly as in the proof of Theorem 11.3.

Here we use a shortcut. We <u>could</u> choose α in such a way that its norm depends only on the "allowed quantities": e.g. $\alpha(z) = \dfrac{K(z,0)}{K(0,0)}$. Then it is clear that we could calculate the inverse of T from (1), and use Proposition 11.2, as in the proof of Theorem 11.3, to expand $K_1(\xi,\zeta)$, which as we have seen is the inner product between the functionals given by evaluation at ξ and ζ in H_1^2. Thus we see that $\mathrm{Pert}(K_E(\xi,\zeta);0)$ exists and that the error terms in the implied expansion of $K_1(\xi,\zeta)$ depend only on the "allowed quantities". However, for the actual computation of the value of $\mathrm{Pert}(K_E(\xi,\zeta);0)$, <u>any</u> function α analytic on $\overline{\Omega}$ and satisfying $\alpha(0) = 1$ would do: we need not worry about the size of its norm.

For ease in calculation we shall choose α so that $\alpha(\xi) = \alpha(\zeta) = 0$.

The top left-hand block of the inverse of T is:

(2)
$$S = \left[\delta_{mn} - 2\pi\epsilon\overline{u_m(0)}u_n(0) + \frac{\epsilon}{2\pi}\int_{\partial\Omega}\frac{\overline{u_m(z)}\alpha(z)ds}{z}\int_{\partial\Omega}\frac{u_n(z)\overline{\alpha(z)}ds}{\overline{z}}\right] + O(\epsilon^2).$$

Since by assumption $\alpha(\xi) = \alpha(\zeta) = 0$, Proposition 11.2 says that:

$$\begin{aligned} K_1(\xi,\zeta) &= \sum S_{mn}u_m(\xi)\overline{u_n(\zeta)} \\ &= K(\xi,\zeta) - 2\pi\epsilon\overline{K(0,\xi)}K(0,\zeta) + \frac{\epsilon}{2\pi}\int_{\partial\Omega}\frac{\overline{K(z,\xi)}\alpha(z)ds}{z}\overline{\int_{\partial\Omega}\frac{\overline{K(z,\zeta)}\alpha(z)ds}{z}} + O(\epsilon^2). \end{aligned}$$

Replacing $\overline{K(z,\xi)}ds$ by $\frac{1}{i}L(z,\xi)dz$ and using the residue theorem shows that the first integral is $2\pi\left(\frac{1}{2\pi}\frac{\alpha(\xi)}{\xi} + L(0,\xi)\alpha(0)\right) = 2\pi L(0,\xi)$ since $\alpha(\xi) = 0$ and $\alpha(0) = 1$. Similarly the second integral is $2\pi L(0,\zeta)$. Therefore:

$$\operatorname{Pert}(K_E(\xi,\zeta);0) = 2\pi(L(0,\xi)\overline{L(0,\zeta)} - \overline{K(0,\xi)}K(0,\zeta)).$$

12.2 <u>Theorem</u> Let E be any compact subset of S^2. Let ξ, ζ and η be distinct points of $S^2 - E$, and let $\eta \neq \infty$. Then:

$$\operatorname{Pert}(L_E(\xi,\zeta);\eta) = 2\pi(L_E(\eta,\xi)\overline{K_E(\eta,\zeta)} - \overline{K_E(\eta,\xi)}L_E(\eta,\zeta)).$$

<u>Proof</u> We assume that ξ, ζ and η are all in the same component Ω of $S^2 - E$, and that Ω is a bounded domain of type G. We take $\eta = 0$. Write $\Omega_1 = \Omega - \overline{D}(0;\epsilon)$. Let H^{2+} and H_1^{2+} be the spaces obtained from $H^2(\Omega)$ and $H^2(\Omega_1)$ respectively by adjoining the function $\frac{1}{z-\zeta}$ in the manner of §8. Denote the Szegö kernels of Ω and Ω_1 by K and K_1 respectively, and the corresponding co-kernels by L and L_1.

Choose an orthonormal basis $\{u_n\}$ for H^{2+}. Let α be a function, analytic on $\overline{\Omega}$, satisfying $\alpha(0) = 1$. For $n \geq 1$ write:

$$v_n(z) = \frac{\epsilon^{n-\frac{1}{2}}}{\sqrt{2\pi}}\frac{\alpha(z)}{z^n}.$$

The matrix T of inner products is given as before by (1), and the top left-hand block S of its inverse is given by (2). We saw in §8 that $\frac{L(\xi,\zeta)}{2\pi K(\zeta,\zeta)}$ is the inner product between evaluation at ξ and the functional "res". Hence $\mathrm{Pert}\left(\frac{L(\xi,\zeta)}{2\pi K(\zeta,\zeta)};0\right)$ exists (compare the last proof). To compute its value we may assume that $\alpha(\xi) = 0$. Then:

$$\frac{L_1(\xi,\zeta)}{2\pi K_1(\zeta,\zeta)} = \sum S_{mn} u_m(\xi)\overline{\mathrm{res}(u_n)}$$

$$= \frac{L(\xi,\zeta)}{2\pi K(\zeta,\zeta)} - 2\pi\epsilon \sum_m u_m(\xi)\overline{u_m(0)}\sum_n u_n(0)\overline{\mathrm{res}(u_n)}$$

$$+ \frac{\epsilon}{2\pi}\int_{\partial\Omega}\frac{\sum u_m(\xi)\overline{u_m(z)}\alpha(z)ds}{z}\int_{\partial\Omega}\frac{\sum u_n(z)\overline{\mathrm{res}(u_n)}\,\overline{\alpha(z)}ds}{\bar{z}} + O(\epsilon^2).$$

We know that $\sum u_m(z)\overline{\mathrm{res}(u_m)} = \frac{L(z,\zeta)}{2\pi K(\zeta,\zeta)}$ and that $\sum u_m(z_1)\overline{u_m(z_2)} = K(z_1,z_2) + \frac{L(z_1,\zeta)\overline{L(z_2,\zeta)}}{K(\zeta,\zeta)}$. We now substitute these in the last expression, and evaluate the integrals by means of the relations $\overline{L(z,\zeta)}ds = \frac{1}{i}K(z,\zeta)dz$ and $\overline{K(z,\xi)}ds = \frac{1}{i}L(z,\xi)dz$ and the residue theorem. Thus:

$$\mathrm{Pert}\left(\frac{L_E(\xi,\zeta)}{2\pi K_E(\zeta,\zeta)};0\right) = \frac{1}{2\pi}\int_{\partial\Omega}\frac{\left(\overline{K(z,\xi)} + \frac{L(\xi,\zeta)\overline{L(z,\zeta)}}{K(\zeta,\zeta)}\right)\alpha(z)ds}{z}\overline{\int_{\partial\Omega}\frac{\frac{\overline{L(z,\zeta)}}{2\pi K(\zeta,\zeta)}\alpha(z)ds}{z}}$$

$$-2\pi\left(K(\xi,0) + \frac{L(\xi,\zeta)\overline{L(0,\zeta)}}{K(\zeta,\zeta)}\right)\frac{L(0,\zeta)}{2\pi K(\zeta,\zeta)}$$

$$= \frac{1}{2\pi}\left(2\pi L(0,\xi) + \frac{2\pi L(\xi,\zeta)}{K(\zeta,\zeta)}K(0,\zeta)\right)\frac{\overline{K(0,\zeta)}}{K(\zeta,\zeta)} - 2\pi\left(K(\xi,0) + \frac{L(\xi,\zeta)\overline{L(0,\zeta)}}{K(\zeta,\zeta)}\right)\frac{L(0,\zeta)}{2\pi K(\zeta,\zeta)}.$$

Also we know that $\mathrm{Pert}(2\pi K_E(\zeta,\zeta);0) = 4\pi^2(|L(0,\zeta)|^2 - |K(0,\zeta)|^2)$. Hence by the "product rule" of elementary calculus:

$$\mathrm{Pert}(L_E(\xi,\zeta);0) = 2\pi K(\zeta,\zeta)\mathrm{Pert}\left(\frac{L_E(\xi,\zeta)}{2\pi K_E(\zeta,\zeta)};0\right) + \frac{L(\xi,\zeta)}{2\pi K(\zeta,\zeta)}\mathrm{Pert}(2\pi K_E(\zeta,\zeta);0)$$

$$= 2\pi(L(0,\xi)\overline{K(0,\zeta)} - \overline{K(0,\xi)}L(0,\zeta))$$

as required.

12.3 Theorem Let E be any compact subset of $\underset{\sim}{C}$ with $\gamma(E) > 0$. Let z and ζ be distinct points of $\underset{\sim}{C} - E$. Then:

$$\mathrm{Pert}(\gamma(E)K_E^+(z,\infty)f_E(z);\eta) = 2\pi\gamma(E)(-L_E(\eta,z)\overline{K_E^+(\eta,\infty)} - \overline{K_E(\eta,z)}K_E^+(\eta,\infty)f_E(\eta))$$
$$\mathrm{Pert}(-\gamma(E)K_E^+(z,\infty)\,;\,\eta) = 2\pi\gamma(E)(L_E(\eta,z)\overline{K_E^+(\eta,\infty)}\,\overline{f_E(\eta)} + \overline{K_E(\eta,z)}K_E^+(\eta,\infty)).$$

The proof is along exactly the same lines as the proofs of Theorem 12.1 and Theorem 12.2, and is omitted. The result is precisely what is expected in view of the discussion in §10.

§13 Perturbation and the Subadditivity Problem

One of the most interesting unsolved problems about analytic capacity is whether γ is subadditive, i.e. $\gamma(E \cup F) \leq \gamma(E) + \gamma(F)$ for all compact plane sets E and F, or possibly even strongly subadditive, i.e. $\gamma(E \cup F) \leq \gamma(E) + \gamma(F) - \gamma(E \cap F)$ for all compact plane sets E and F. The relation $\gamma(E \cup \overline{D}(\eta;\epsilon)) = \gamma(E) + a_E(\eta)\epsilon + O(\epsilon^2)$ shows that if γ is subadditive then $a_E \leq 1$ for all compact plane sets E. In Chapters IV and V we shall show that $a_E \leq 1$ for certain classes of sets E. Meantime we show that the statement that $a_E \leq 1$ whenever E is connected reduces to one of the distortion theorems of classical complex analysis.

13.1 Theorem Let E be a compact connected plane set. Then $a_E \leq 1$.

Proof If E is a singleton then $a_E = 1$. If E is not a singleton then we may assume, after a suitable scaling, that $\gamma(E) = 1$. Let ϕ be the conformal map of $S^2 - \overline{D}$ onto $\Omega(E)$ which takes ∞ to ∞ and has positive derivative at ∞: so that, near ∞, $\phi(z) = z + a_0 + a_1/z + \ldots\ldots$ Whenever $|z| > 1$:

$$a_E(\phi(z)) = 4\pi^2|K_E^+(\phi(z),\infty)|^2\{1 - |f_E(\phi(z))|^2\}$$
$$= 4\pi^2|\phi'(z)|^{-1}|K_{\overline{D}}^+(z,\infty)|^2\{1 - |f_{\overline{D}}(z)|^2\}$$

by Proposition 2.9 and formula (5) of §10

$$= |\phi'(z)|^{-1}\{1 - \frac{1}{|z|^2}\}.$$

The Koebe-Bieberbach distortion theorem ([4], p. 185) says precisely that this is at most 1.

Let E be a compact subset of S^2, and let z, ζ and a be distinct points of $\underset{\sim}{C} - E$. Define:

$$b_E(z,\zeta,a) = \mathrm{Pert}\{\mathrm{Pert}(2\pi K_E(a,a);z)\ ;\ \zeta\}.$$

Thus by Theorem 12.1, Theorem 12.2 and the "product rule":

$$b_E(z,\zeta,a) = 4\pi^2\mathrm{Pert}\{L_E(z,a)\overline{L_E(z,a)} - K_E(z,a)\overline{K_E(z,a)}\ ;\ \zeta\}$$

$$= 8\pi^3\overline{L_E(z,a)}\{L_E(\zeta,z)\overline{K_E(\zeta,a)} - \overline{K_E(\zeta,z)}L_E(\zeta,a)\} + \text{complex conjugate}$$
$$- 8\pi^3\overline{K_E(z,a)}\{L_E(\zeta,z)\overline{L_E(\zeta,a)} - \overline{K_E(\zeta,z)}K_E(\zeta,a)\} + \text{complex conjugate}$$

$$\begin{aligned} &= 16\pi^3\ \mathrm{Re}\{K_E(z,\zeta)K_E(\zeta,a)K_E(a,z) + K_E(z,\zeta)L_E(\zeta,a)\overline{L_E(a,z)} \\ &\qquad + \overline{L_E(z,\zeta)}K_E(\zeta,a)L_E(a,z) + L_E(z,\zeta)\overline{L_E(\zeta,a)}K_E(a,z)\}. \end{aligned} \tag{1}$$

Observe the symmetry in z, ζ and a. If z, ζ and a are distinct points of $S^2 - E$ and one of them is ∞ then we write $b_E(z,\zeta,a) = 0$. If E_1 and E_2 are compact subsets of S^2 and if ϕ maps a component Ω of $S^2 - E_2$ onto a component of $S^2 - E_1$, then it follows from (1) and Theorem 9.5 that:

$$b_{E_2}(z,\zeta,a) = |\phi'(z)||\phi'(\zeta)||\phi'(a)|b_{E_1}(\phi(z),\phi(\zeta),\phi(a))$$

for all distinct z, ζ, $a \in \Omega$. Hence the sign of $b_E(z,\zeta,a)$ is invariant under conformal mapping.

For computational purposes we shall sacrifice the symmetry of $b_E(z,\zeta,a)$ by taking ∞ as reference point instead of a. Let E be a compact subset of $\underset{\sim}{C}$, and let z and ζ be distinct points of $\underset{\sim}{C} - E$. We define:

$$b_E(z,\zeta) = \mathrm{Pert}\{\mathrm{Pert}(\gamma(E);z);\zeta\} = \mathrm{Pert}\{a_E(z);\zeta\}.$$

Since $a_E(z) = 4\pi^2\gamma(E)^2|K_E^+(z,\infty)|^2\{1 - |f_E(z)|^2\}$, Theorem 12.3 gives:

$$\begin{aligned} b_E(z,\zeta) = 16\pi^3\gamma(E)^2\mathrm{Re}\{&K_E(z,\zeta)K_E^+(\zeta,\infty)f_E(\zeta)\overline{K_E^+(z,\infty)}\,\overline{f_E(z)} - K_E(z,\zeta)K_E^+(\zeta,\infty)\overline{K_E^+(z,\infty)} \\ (2) \qquad &+ L_E(z,\zeta)\overline{K_E^+(\zeta,\infty)}\,\overline{f_E(\zeta)}\,\overline{K_E^+(z,\infty)} - L_E(z,\zeta)\overline{K_E^+(\zeta,\infty)}\,\overline{K_E^+(z,\infty)}\,\overline{f_E(z)}\}. \end{aligned}$$

Formulae (3) and (4) of §10 show that if E_1 is a compact subset of S^2, and E_2 is a compact subset of $\underset{\sim}{C}$, and ϕ maps $\Omega(E_2)$ conformally onto a component of $S^2 - E_1$, and $\phi(\infty) \neq \infty$, then:

$$b_{E_2}(z,\zeta) = |\phi'(z)||\phi'(\zeta)||\phi'(\infty)|b_{E_1}(\phi(z),\phi(\zeta),\phi(\infty))$$

for all distinct $z, \zeta \in \Omega(E)$. So there is no loss of generality in considering $b_E(z,\zeta)$ instead of $b_E(z,\zeta,a)$. Also if E_1 and E_2 are compact subsets of $\underset{\sim}{C}$, if ϕ maps $\Omega(E_2)$ conformally onto $\Omega(E_1)$, and if $\phi(\infty) = \infty$, then:

$$b_{E_2}(z,\zeta) = |\phi'(z)||\phi'(\zeta)||\phi'(\infty)|b_{E_1}(\phi(z),\phi(\zeta))$$

for all distinct $z, \zeta \in \Omega(E)$. Hence the sign of $b_E(z,\zeta)$ is invariant under any conformal mapping which takes ∞ to ∞.

The reason for our interest in the sign of b_E is as follows. Suppose that γ is strongly subadditive. Let E and F be compact subsets of $\underset{\sim}{C}$, with $E \subset F$, and let $\eta \in \Omega(F)$. Let $0 < \epsilon < d(\eta;F)$. Applying the hypothesis of strong subadditivity to the sets F and $E \cup \overline{D}(\eta;\epsilon)$, we have:

$$\gamma(F \cup \bar{D}(\eta;\epsilon)) \leqslant \gamma(F) + \gamma(E \cup \bar{D}(\eta;\epsilon)) - \gamma(E);$$

$$\frac{1}{\epsilon}(\gamma(F \cup \bar{D}(\eta;\epsilon)) - \gamma(F)) \leqslant \frac{1}{\epsilon}(\gamma(E \cup \bar{D}(\eta;\epsilon)) - \gamma(E)).$$

Letting $\epsilon \to 0$ gives $a_F(\eta) \leqslant a_E(\eta)$. Therefore strong subadditivity of γ would imply that a_E is a decreasing function of E. That, in turn, would obviously imply that $b_E(z,\zeta) = \mathrm{Pert}\{a_E(z);\zeta\}$ is non-positive. The connection, however, runs deeper than that. In Chapter V we shall develop a technique which allows us, in certain circumstances, to reverse the above trivial implications and to use partial results about non-positivity of b_E to deduce partial results about strong subadditivity.

In Chapter IV we shall prove that $b_E \leqslant 0$ for certain types of set E. But once again the case when E is connected is straightforward.

13.2 <u>Theorem</u> Let $z, \zeta \in \mathbb{C}$, $|z| > 1$, $|\zeta| > 1$, $z \neq \zeta$. Then:

$$b_{\bar{D}}(z,\zeta) = \frac{-|z\bar{\zeta} - \bar{z}\zeta|^2(|z|^2 - 1)(|\zeta|^2 - 1)}{|z|^2|\zeta|^2|z\bar{\zeta} - 1|^2|z - \zeta|^2}.$$

<u>Proof</u> Simply substitute the expressions in the last paragraph of §10 into (2). Thus:

$$\begin{aligned} b_{\bar{D}}(z,\zeta) &= 2\,\mathrm{Re}\left(\frac{1}{(z\bar{\zeta} - 1)\bar{z}\zeta} - \frac{1}{z\bar{\zeta} - 1} + \frac{1}{(z - \zeta)\bar{\zeta}} - \frac{1}{(z - \zeta)\bar{z}}\right) \\ &= 2\,\mathrm{Re}\,\frac{(\bar{z}\zeta - z\bar{\zeta})(|\zeta|^2 - 1)}{(z\bar{\zeta} - 1)(z - \zeta)\bar{z}|\zeta|^2} \\ &= \frac{(\bar{z}\zeta - z\bar{\zeta})(|\zeta|^2 - 1)}{|\zeta|^2}\left(\frac{1}{(z\bar{\zeta} - 1)(z - \zeta)\bar{z}} - \frac{1}{(\bar{z}\zeta - 1)(\bar{z} - \bar{\zeta})z}\right) \\ &= \frac{-|z\bar{\zeta} - \bar{z}\zeta|\,(|z|^2 - 1)(|\zeta|^2 - 1)}{|z|^2|\zeta|^2|z\bar{\zeta} - 1|^2|z - \zeta|^2}. \end{aligned}$$

It is obvious from Theorem 13.2 that $b_{\bar{D}} \leqslant 0$ and that $b_{\bar{D}}(z,\zeta) = 0$ if and only if z and ζ lie on the same straight line through 0. $b_{\bar{D}}(z,\zeta)$, as a function of z, does not extend to be continuous at $z = \zeta$. Elementary trigonometry shows that $b_{\overline{D}}$ is bounded below and that $\inf b_{\overline{D}} = -4$.

13.3 <u>Corollary</u> Let E be a compact connected subset of S^2. Then $b_E \leqslant 0$.

<u>Proof</u> The sign of b_E is invariant under conformal mapping.

CHAPTER IV

COMPACT SUBSETS OF $\underset{\sim}{R}$

In this chapter we derive explicit formulae for the domain functions of a domain whose complement is contained in $\underset{\sim}{R}$, and we deduce properties of such domains and of domains conformally equivalent to them. Most of the material in this chapter is new. §15 overlaps with a paper of Ch. Pommerenke [18], but the approaches are different.

§14 An Approach via Harmonic Functions

In this section and the next we shall use the concept of harmonic measure. To introduce the harmonic measure of an arbitrary domain would be complicated and would assume a knowledge of the Dirichlet problem; and so we shall merely adapt, as little as is necessary for our purpose, the well-known theory of the Poisson integral for the unit disc.

Let Ω be a Jordan domain in S^2, i.e. a domain in S^2 whose boundary is a Jordan curve in S^2. Let $\zeta \in \Omega$. Choose any conformal map ϕ of Ω onto D such that $\phi(\zeta) = 0$. ϕ extends to a homeomorphism of $\overline{\Omega}$ onto $\overline{D}$ ([15], theorem 17.5.3). For each Borel subset E of $\partial\Omega$, the <u>harmonic measure</u> of E (at the point ζ, w.r.t. the domain Ω) is defined as the arc length of $\phi(E)$, divided by 2π. Harmonic measure is therefore a probability measure on $\partial\Omega$. Since ϕ is unique up to a rotation, the definition is independent of the choice of ϕ. If Γ is an analytic arc contained in $\partial\Omega$, then ϕ continues analytically across Γ, so that, on Γ, harmonic measure and arc length are equivalent (mutually absolutely continuous). If f is any bounded Borel function on $\partial\Omega$, then the <u>harmonic extension</u> of f to Ω is the function on Ω whose value at each point $\zeta \in \Omega$ is the integral of f w.r.t. harmonic measure at ζ. If Ω_1 and Ω_2 are Jordan domains in S^2, and ϕ maps Ω_2 conformally onto Ω_1, then it is clear from the definition that the harmonic measure of any Borel set $E \subset \partial\Omega_2$ at a point $\zeta \in \Omega_2$ w.r.t. Ω_2 is equal to the harmonic measure of $\phi(E)$ at $\phi(\zeta)$ w.r.t. Ω_1.

We can compute harmonic measure explicitly for the unit disc D. Let $\zeta \in D$. Then $\phi(z) = \dfrac{z-\zeta}{1-\bar{\zeta}z}$ maps D conformally onto D, and $\phi(\zeta) = 0$. At any point $z \in \partial D$, $|\phi'(z)| = \dfrac{1-|\zeta|^2}{|z-\zeta|^2} = \mathrm{Re}\dfrac{z+\zeta}{z-\zeta}$. So harmonic measure μ at ζ is $d\mu(z) = \dfrac{ds}{2\pi}\mathrm{Re}\,\dfrac{z+\zeta}{z-\zeta}$. Hence if f is any bounded Borel function on the unit circle, then its harmonic extension is given by:

$$Pf(\zeta) = \frac{1}{2\pi}\int_{\partial D} f(z)\,\mathrm{Re}\frac{z+\zeta}{z-\zeta}\,ds.$$

This is the Poisson integral for f. It is well known that the mapping $f \to Pf$ is a linear isometry of $L^\infty(\partial D)$ onto the Banach space of all bounded harmonic functions on D with the supremum norm. If $f \in L^\infty(\partial D)$ then for almost every $z \in \partial D$ the non-tangential limit of Pf at z exists and equals $f(z)$. If in addition f is continuous at some point z then Pf extends to be continuous at z and $Pf(z) = f(z)$. See [16], p. 38. Observe that the harmonic measures at any two points of D are equivalent. The same result follows for an arbitrary Jordan domain by conformal mapping.

It will be convenient to extend slightly the notion of the non-tangential limit of a function defined on the unit disc. Let Ω

be a Jordan domain, g a function defined on Ω, $z_0 \in \partial\Omega$. If $a \in \mathbb{C}$, and if $g(z) \to a$ as $z \to z_0$ along any curve in Ω whose image, under a conformal map of Ω onto D, approaches ∂D non-tangentially, then we say that a is <u>the "non-tangential limit" of g at</u> z_0. Clearly then "non-tangential limits" are invariant under conformal mapping of a Jordan domain onto a Jordan domain. By applying a conformal mapping to the results quoted in the last paragraph we now obtain the following theorem.

14.1 <u>Theorem</u> Let Ω be a Jordan domain in S^2, and let μ be harmonic measure at any point of Ω. Then:

(a) The harmonic extension $f \to Pf$ is a linear isometry of $L^\infty(\mu)$ onto the Banach space of all bounded harmonic functions on Ω with the supremum norm.

(b) If $f \in L^\infty(\mu)$ then for almost every (μ) $z \in \partial\Omega$ the "non-tangential limit" of Pf at z exists and equals $f(z)$.

(c) If $f \in L^\infty(\mu)$ and f is continuous at a point $z \in \partial\Omega$ then Pf extends to be continuous at z and $Pf(z) = f(z)$.

One domain we shall be concerned with is the lower half plane. The conformal mapping $z \to \dfrac{z-\zeta}{z-\overline{\zeta}}$ of the lower half plane onto D shows that harmonic measure at ζ is given by $\dfrac{dx}{\pi}\operatorname{Im}\dfrac{1}{\zeta - x}$ along the real axis.

We shall say that a subset of S^2 is <u>symmetric</u> if it is invariant under complex conjugation. We shall say that a function g defined on a symmetric set Ω in S^2 is <u>symmetric</u> if $g(z) = \overline{g(\overline{z})}$ for all $z \in \Omega$. If $E \subset S^2$ then the intersections of E with the upper and lower half planes will be denoted by E_U and E_L respectively. $\overline{\mathbb{R}}$ of course denotes the closure of $\mathbb{R}$ in S^2. We shall denote by I the class of all compact subsets of $\underset{\sim}{C}$ all of whose components are symmetric.

14.2 Proposition Let Ω be a symmetric domain in S^2, and let K and L be its Szegö kernel and co-kernel respectively. Then $K(z,\zeta) = \overline{K(\bar{z},\bar{\zeta})}$ $(z,\zeta \in \Omega)$ and $L(z,\zeta) = \overline{L(\bar{z},\bar{\zeta})}$ $(z,\zeta \in \Omega,$ $z \neq \zeta)$.

Proof By Theorem 9.1 and Theorem 9.4 we may assume that $\Omega \in G$. Let $\zeta \in \Omega$, $\zeta \neq \infty$. Write $K_\zeta(z) = \overline{K(\bar{z},\bar{\zeta})}$ and $L_\zeta(z) = \overline{L(\bar{z},\bar{\zeta})}$. Clearly K_ζ is analytic in $\bar{\Omega}$, L_ζ is analytic in $\bar{\Omega}$ except for a simple pole at ζ of residue $\frac{1}{2\pi}$, and $\bar{K}_\zeta ds = \mp L_\zeta dz$ round $\partial\Omega$ according as ∞ is in Ω or not. We saw in §8 that these properties imply that $K_\zeta(z) = K(z,\zeta)$ and $L_\zeta(z) = L(z,\zeta)$.

The Ahlfors function and the Szegö kernel at ∞ of a compact symmetric subset of $\underset{\sim}{C}$ are symmetric. This follows from Proposition 14.2 or can be proved directly.

14.3 Proposition Let $E \in I$. Then there is a symmetric conformal mapping ϕ of $\Omega(E)$ onto a domain in S^2 whose complement is contained in $\underset{\sim}{R}$, such that $\phi(\infty) = \infty$, $\phi'(\infty) = 1$, $\phi(\Omega(E)_U) = \underset{\sim}{C}_U$ and $\phi(\Omega(E)_L) = \underset{\sim}{C}_L$.

Proof The Riemann mapping theorem gives a conformal map g of $\Omega(E)_U$ onto $\underset{\sim}{C}_U$. Continue g to $\Omega(E)$ by Schwarz reflection. The required mapping is obtained by composing g with a suitable linear fractional transformation.

We shall make use of the conformal mapping $\omega \to \dfrac{e^{\pi\omega/2} - 1}{e^{\pi\omega/2} + 1}$ of the strip $\{\omega : -1 < \operatorname{Im}\omega < 1\}$ onto D. Note that the mapping is symmetric, that it maps the upper and lower halves of the strip onto D_U and D_L respectively, and that its derivative at 0 is $\frac{\pi}{4}$.

Symmetry considerations give a useful alternative formulation of the Ahlfors extremal problem for a set in I. Let $E \in I$. Denote by $B(E)$ the real Banach space of all bounded real-valued harmonic functions on $\Omega(E)_L$ which extend continuously to be 0 on $\Omega(E) \cap \overline{\underset{\sim}{R}}$. Let $u \in B(E)$. Since $\Omega(E)_L$ is simply-connected, u is the imaginary part of some analytic function h on $\Omega(E)_L$. h continues by Schwarz reflection to be a symmetric analytic function on the whole of $\Omega(E)$. $h(\infty) \in \underset{\sim}{R}$, and we make the construction unique by stipulating that $h(\infty) = 0$. We define $\Psi(u) = \frac{\pi}{4} h'(\infty)$. $\Psi(u) \in \underset{\sim}{R}$ since $h(x)$ is real for real x. Clearly Ψ is a bounded linear functional on $B(E)$. If $u \geq 0$ on $\Omega(E)_L$ then $\Psi(u) \geq 0$. Now assume that u is in the closed unit ball $B(E)_1$ of $B(E)$. Then h maps $\Omega(E)$ into the strip $\{\omega : -1 < \operatorname{Im} \omega < 1\}$, and $h(\infty) = 0$. Thus $f = \dfrac{e^{\pi h/2} - 1}{e^{\pi h/2} + 1}$ is a symmetric function admissible for E. This process is reversible: for if f is any symmetric function admissible for E then $h = \frac{2}{\pi} \log\left(\frac{1+f}{1-f}\right)$ is a symmetric analytic function of $\Omega(E)$ into the strip $\{\omega : -1 < \operatorname{Im} \omega < 1\}$ and so $\operatorname{Im} h$ is a harmonic function u on $\Omega(E)_L$, bounded by 1, which extends continuously to be 0 on $\Omega(E) \cap \overline{\underset{\sim}{R}}$. Thus we have a natural one-one correspondence between $B(E)_1$ and the set of symmetric functions admissible for E. Since $\Psi(u) = \frac{\pi}{4} h'(\infty) = f'(\infty)$, and since the Ahlfors function of E is symmetric, the extremal problem for E is just the problem of maximising $\Psi(u)$, where $u \in B(E)_1$. Theorem 2.3 now says that there is a unique function $u_E \in B(E)_1$ which maximises $\Psi(u)$, and that $\Psi(u_E) = \gamma(E)$.

14.4 <u>Theorem</u> Let E be a compact subset of $\underset{\sim}{R}$. Then:

(a) $B(E)_1$ consists precisely of the harmonic extensions to $\underset{\sim}{C}_L$ of the real Borel functions on E bounded by 1;

(b) $u_E(\zeta) = \frac{1}{\pi}\int_E \mathrm{Im}\left(\frac{1}{\zeta - x}\right) dx;$

(c) $\gamma(E) = \frac{1}{4}\ell(E)$.

<u>Proof</u> (a) is an immediate consequence of Theorem 14.1. If $u \in B(E)_1$ then, by (a), $u(\zeta) = \frac{1}{\pi}\int \mathrm{Im}\left(\frac{1}{\zeta - x}\right)\sigma(x)dx$ for some real Borel function σ on E bounded by 1. Near ∞ this is the imaginary part of the analytic function $h(\zeta) = \frac{1}{\pi}\int \frac{\sigma(x)dx}{\zeta - x}$, so that by definition $\Psi(u) = \frac{\pi}{4}\,h'(\infty) = \frac{1}{4}\int \sigma(x)dx$. Hence to maximise $\Psi(u)$ we choose σ to be identically 1 on E. This gives (b) and (c).

14.5 <u>Corollary</u> Let $E \in I$. Then $u_E > 0$ in $\Omega(E)_L$.

14.6 <u>Corollary</u> Let $E \in I$. Write $V = \Omega(E)_L$, and suppose that V is a Jordan domain. Then u_E is the unique bounded harmonic function on V whose "non-tangential limit" is almost everywhere (w.r.t. harmonic measure) 0 on $\Omega(E) \cap \overline{\underset{\sim}{R}}$ and 1 elsewhere on ∂V.

<u>Proofs</u> If $E \subset \underset{\sim}{R}$ then these are immediate from Theorem 14.4. The results follow for all E by Proposition 14.3.

In the more usual notation, Theorem 14.4 says that the symmetric functions admissible for E are precisely the functions $\frac{e^{\pi h/2} - 1}{e^{\pi h/2} + 1}$ where $h(\zeta) = \frac{1}{\pi}\int \frac{\sigma(x)dx}{\zeta - x}$ for some real Borel function σ on E bounded by 1, and that $f_E = \frac{e^{\pi h/2} - 1}{e^{\pi h/2} + 1}$, where $h(\zeta) = \frac{1}{\pi}\int \frac{dx}{\zeta - x}$. Corollary 14.5 says that if $E \in I$ then f_E maps $\Omega(E)_L$ into D_U and $\Omega(E)_U$ into D_L. Corollary 14.6 says that, in the stated conditions, $f_E|_V$ is the unique analytic function in V whose range lies in D_U and whose non-tangential limits are almost everywhere (w.r.t. harmonic measure) real on $\Omega(E) \cap \overline{\underset{\sim}{R}}$ and of modulus 1 elsewhere on ∂V.

As an application of the above machinery we now prove an easy subadditivity theorem. Strangely, it does not seem to have been spotted before, although Pommerenke [18] and Shirokov [20] and [21] were working in the same circle of ideas.

14.7 <u>Theorem</u> Let $E, F \in I$. Then $\gamma(E \cup F) \leq \gamma(E) + \gamma(F)$.

<u>Proof</u> We may assume that the outer boundaries of E and F are unions of finitely many pairwise disjoint analytic symmetric Jordan curves. Write $V = \Omega(E \cup F)_L$. Since the boundaries of $\Omega(E)_L$, $\Omega(F)_L$ and V are finite unions of analytic arcs, harmonic measure is equivalent to arc length in each case. On that part of ∂V which lies on $\partial\Omega(E)$, $u_E = 1$ a.e. and $u_F \geq 0$. On that part of ∂V which lies on $\partial\Omega(F)$, $u_E \geq 0$ and $u_F = 1$ a.e. Hence the function $(u_E + u_F)|_V$, which is clearly in $B(E \cup F)$, is 0 on $\Omega(E \cup F) \cap \overline{\underset{\sim}{R}}$ and ≥ 1 a.e. elsewhere on ∂V. We know also that $u_{E \cup F}$ is 0 on $\Omega(E \cup F) \cap \overline{\underset{\sim}{R}}$ and 1 a.e. elsewhere on ∂V. Since the correspondence between real bounded Borel functions on ∂V and their harmonic extensions in V is order-preserving, it follows that $u_{E \cup F} \leq (u_E + u_F)|_V$. Hence $\gamma(E \cup F) = \Psi(u_{E \cup F}) \leq \Psi((u_E + u_F)|_V) = \Psi(u_E) + \Psi(u_F) = \gamma(E) + \gamma(F)$.

§15 <u>Shirokov's Conjecture</u>

Let E be a compact plane set. Let $0 < r < 1$. Write $E_r = S^2 - \{z \in \Omega(E) : |f_E(z)| < r\}$. Clearly E_r is a compact plane set containing E. In fact, every component of E_r must contain a component of E, by the maximum modulus theorem. In [20], N.A.

Shirokov conjectures that $f_{E_r} = f_E/r$ in $\Omega(E_r)$. In this section we shall show that his conjecture is true for a certain class of sets E and false in general.

15.1 <u>Proposition</u> Let E be a compact plane set. Let $0 < r < 1$. Then $\gamma(E_r) \geq \gamma(E)/r$, with equality if and only if $f_{E_r} = f_E/r$ in $\Omega(E_r)$.

<u>Proof</u> $\frac{1}{r} f_E\big|_{\Omega(E_r)}$ is admissible for E_r and so $\gamma(E_r) \geq f_E'(\infty)/r = \gamma(E)/r$. If equality holds then $\frac{1}{r} f_E\big|_{\Omega(E_r)}$ must be the Ahlfors function of E_r by Theorem 2.3.

We denote by J the class of all compact plane sets E such that there is a conformal mapping ϕ which maps $\Omega(E)$ onto a domain in S^2 whose complement is contained in $\underset{\sim}{R}$ and which satisfies $\phi(\infty) = \infty$. $I \subset J$ by Proposition 14.3.

15.2 <u>Proposition</u> Let E be a compact plane set with at most two components. Then $E \in J$.

<u>Proof</u> We can ignore the cases when $\Omega(E)$ has an isolated boundary point. If $\Omega(E)$ is simply-connected then the result follows from the Riemann mapping theorem. If $\Omega(E)$ is doubly-connected then there is a conformal mapping ϕ of $\Omega(E)$ onto a domain in S^2 whose complement is the union of two disjoint closed discs in $\underset{\sim}{C}$ ([4], p. 221). We may suppose that $\phi(\infty) = \infty$ and that the centres of the two discs lie on the real axis. The result follows by Proposition 14.3.

15.3 <u>Theorem</u> Shirokov's conjecture is true for all $E \in J$.

<u>Proof</u> It follows from Proposition 2.9 that if E and F are compact plane sets, if ϕ maps $\Omega(E)$ conformally onto $\Omega(F)$, and if $\phi(\infty) = \infty$, then Shirokov's conjecture holds for E if and only if it holds for F. So we may assume that $E \subset \underset{\sim}{R}$. Let $0 < r < 1$. Write $V = \Omega(E_r)_L$. We shall show that V is a Jordan domain.

Denote by S the set of all $x \in E$ such that $|f_E(x+iy)| \not\to 1$ as $y \to 0-$. By the remarks after Corollary 14.6, S has harmonic measure zero w.r.t. $\underset{\sim}{C}_L$, i.e. zero length. For each $x \in \underset{\sim}{R}$, write $\ell_x = \{x+iy : y < 0\}$. Let $x \in \underset{\sim}{R}$, and suppose that ℓ_x meets ∂V infinitely often. Then there is an infinite collection of points on ℓ_x at which $|f_E| = r$. This collection is bounded since $f_E(\infty) = 0$, and so has a cluster point. As it cannot cluster at any point at which f is analytic, we conclude that $x \in E$ and that it clusters at x. Hence $\liminf_{y \to 0-} |f_E(x+iy)| \leqslant r < 1$, and so $x \in S$. We conclude that the set of points $x \in \underset{\sim}{R}$ such that ℓ_x meets ∂V infinitely often has zero length.

Now let Γ be any path component of $\partial V \cap \underset{\sim}{C}_L$. Let $z_1, z_2 \in \Gamma$. There is a path in Γ joining z_1 and z_2, and this path clearly consists of a union of finitely many analytic arcs on which $|f_E| = r$, joined together by their endpoints. (A join may occur at any point of the path at which $f_E' = 0$.) $f_E(\underset{\sim}{C}_L) \subset D_U$, so that $\arg f_E$ is a well-defined function on $\underset{\sim}{C}_L$. If we traverse the path with V on our left then $\arg f_E$ strictly increases: so $\arg f_E(z_1) \neq \arg f_E(z_2)$. Hence the continuous function $z \to \arg f_E(z)$ is one-one on Γ. By considering the local behaviour of Γ, it is easy to see that it is bicontinuous. It follows that Γ is homeomorphic to an interval, which, again by the local behaviour of

Γ, is open. Say $\Gamma = \tau(]0,1[)$, where τ is a homeomorphism. As $\lambda \to 0+$, $\operatorname{Im} \tau(\lambda) \to 0$. Also, as $\lambda \to 0+$, $\operatorname{Re} \tau(\lambda)$ converges: since if $\operatorname{Re} \tau(\lambda)$ oscillated between a and b $(a < b)$ then Γ would meet ℓ_x infinitely often for all $x \in]a,b[$, contradicting the conclusion of the last paragraph. So, as $\lambda \to 0+$, $\tau(\lambda)$ converges to some point of $\underset{\sim}{R}$. Similarly, as $\lambda \to 1-$, $\tau(\lambda)$ converges to some point of $\underset{\sim}{R}$. Hence $\overline{\Gamma}$ is a Jordan arc with both endpoints on $\underset{\sim}{R}$. We see therefore that ∂V is the union of a countable sequence $\{\Gamma_n\}$ of Jordan arcs with both endpoints on $\underset{\sim}{R}$, and some subset of $\overline{\underset{\sim}{R}}$.

To complete the proof that ∂V is a Jordan curve, all that remains to be shown is that $\operatorname{diam} \Gamma_n \to 0$ as $n \to \infty$. By the <u>width</u> and the <u>height</u> of a plane set we mean the lengths of its projections on the real and imaginary axes respectively. Let $\epsilon > 0$. Since f_E is analytic on the line $\{z : \operatorname{Im} z = -\epsilon\}$, $|f_E| = r$ at only finitely many points of that line. Hence only finitely many of the Γ_n have a height ϵ or larger. Suppose now that infinitely many of the Γ_n have a width ϵ or larger. Since all of the Γ_n are contained in the bounded set E_r, there must be some interval $[c - \frac{1}{2}\epsilon, c]$ which contains the left-hand endpoints of the projections on the real axis of infinitely many of those Γ_n whose width is ϵ or larger. Then, for all $x \in [c, c + \frac{1}{2}\epsilon]$, ℓ_x meets all these Γ_n, which is impossible. So only finitely many of the Γ_n have a width or a height as large as ϵ. That is, $\operatorname{diam} \Gamma_n \to 0$ as $n \to \infty$. It is now easy to see that ∂V must be a Jordan curve. So V is a Jordan domain.

For each n let F_n be the compact connected symmetric set consisting of the arc Γ_n, its reflection in the real axis, and the inside of the Jordan curve formed by these two arcs. Clearly E_r

is contained in the union of E and all the F_n. Let $x \in E - S$. Then for all negative y sufficiently near 0, $x + iy \in E_r$. So $x \in F_n$ for some n. Hence E_r is contained in the union of S and all the F_n.

Consider the function f_E/r defined on V. It is analytic on V and its values lie in D_U. It extends continuously to $\Omega(E_r) \cap \bar{\mathbb{R}}$ and takes real values there. It extends continuously to $\partial V - \bar{\mathbb{R}}$, and has modulus 1 there. The only remaining points of ∂V comprise a set S_1 which, by the last paragraph, is contained in the union of S and the countable set consisting of the endpoints of all the Γ_n, and which therefore has zero length, i.e. harmonic measure zero w.r.t. $\mathbb{C}_L$. Since harmonic measure w.r.t. a domain U increases as U increases ([24], p. 111), S_1 has zero harmonic measure w.r.t. V also. Hence by the remarks after Corollary 14.6, $f_E/r = f_{E_r}$ on V. Equality persists throughout $\Omega(E_r)$ since both sides are analytic.

The above theorem was proved by Shirokov ([21], pp. 79 - 81) in the case when E has at most countably many components. It shows in particular that Shirokov's conjecture holds for any compact plane set with at most two components.

15.4 <u>Lemma</u> Let E be a symmetric compact subset of $\mathbb{C}$. Suppose that $\Omega(E)$ is finitely connected and that one of the components of its boundary is a symmetric Jordan curve Γ. Let a and b $(a < b)$ be the two points at which Γ meets $\mathbb{R}$. Then $f_E(a) = -1$ and $f_E(b) = 1$.

<u>Proof</u> By Corollary 7.2 (c), f_E extends continuously to Γ and has modulus 1 there. We can assume, after a conformal

mapping, that $\Omega(E)$ is bounded by finitely many disjoint analytic Jordan curves. $i\psi_E(b) = 2\pi\gamma(E)^2 K_E^+(b,\infty) > 0$ by the remarks after Proposition 14.2. At b, $f_E(z)\psi_E(z)dz > 0$ and dz points upwards; so $f_E(b) > 0$. Therefore $f_E(b) = 1$. $f_E(a) = -1$ similarly.

15.5 <u>Theorem</u> There exists a compact plane set E, with three components, for which Shirokov's conjecture fails.

<u>Proof</u> Write $F = \{iy : -4 \leqslant y \leqslant -1 \text{ or } 1 \leqslant y \leqslant 4\}$. We can compute the Ahlfors function of $[-4,-1]\cup[1,4]$ by the formula given after Corollary 14.6 and obtain f_F from it by rotation. This gives $f_F = \frac{1}{i}\frac{g^{\frac{1}{2}}-1}{g^{\frac{1}{2}}+1}$, where $g(\zeta) = \frac{(\zeta-i)(\zeta+4i)}{(\zeta-4i)(\zeta+i)}$. f_F' vanishes when and only when g' vanishes, i.e. at 2 and -2. Elementary computation gives $f_F(2) = \frac{1}{3}$ and $f_F(-2) = -\frac{1}{3}$. Of course f_F takes real values on $\underset{\sim}{R}$ (since F is symmetric). So as x increases from $-\infty$ to -2, $f_F(x)$ decreases from 0 to $-\frac{1}{3}$; as x increases from -2 to 2, $f_F(x)$ increases from $-\frac{1}{3}$ to $\frac{1}{3}$; and as x increases from 2 to ∞, $f_F(x)$ decreases from $\frac{1}{3}$ to 0. $\gamma(F) = \ell(F)/4 = \frac{3}{2}$.

If η and ζ are distinct points of $\underset{\sim}{R}$, then $K_F(\eta,\zeta)$, $L_F(\eta,\zeta)$ and $K_F^+(\eta,\infty)$ are real by Proposition 14.2. If we fix $\zeta \in \underset{\sim}{R}$, then $L_F(\eta,\zeta) = \frac{1}{2\pi(\eta-\zeta)} + \ldots\ldots$ is positive for all η greater than ζ and sufficiently near ζ; since L_F never vanishes, it must be positive for all $\eta > \zeta$. $K^+(\eta,\infty) > 0$ for all $\eta \in \underset{\sim}{R}$, since it never vanishes and is positive when $\eta = \infty$. By Proposition 3.3 (a), $f_F(9) \leqslant \frac{\gamma(F)}{d(9,F)} < \frac{3/2}{9} = \frac{1}{6}$, $f_F(6) < \frac{1}{4}$, and $|f_F| \leqslant \frac{1}{4}$ on the line $\ell = \{z : \operatorname{Re} z = 6\}$.

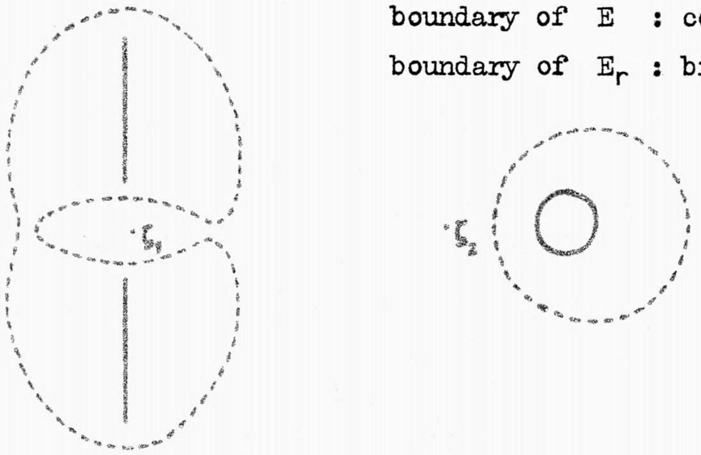
Let ζ be 2 or -2. By Theorem 12.3:

$$\mathrm{Pert}(f_F(\zeta);9) = \mathrm{Pert}\left(\frac{\gamma(F)K_F^+(\zeta,\infty)f_F(\zeta)}{\gamma(F)K_F^+(\zeta,\infty)};9\right)$$
$$= \frac{2\pi K_F^+(9,\infty)L_F(9,\zeta)}{K_F^+(\zeta,\infty)}\left(-1 - \frac{K_F(9,\zeta)}{L_F(9,\zeta)}f_F(9) + f_F(\zeta)f_F(9) + f_F(\zeta)\frac{K_F(9,\zeta)}{L_F(9,\zeta)}\right)$$
$$< 0.$$

(The common factor is positive, and the bracketed expression is negative since its second, third and fourth terms are bounded in magnitude by $1.\frac{1}{6}$, $\frac{1}{3}.\frac{1}{6}$ and $\frac{1}{3}.1$ respectively.)

Now let $E = F \cup \overline{D}(9;\epsilon)$, where ϵ is a positive number chosen to be so small that $f_E(-2) < -\frac{1}{3}$, $f_E < \frac{1}{3}$ on $]-\infty,6]$, $f_E(2) > \frac{1}{4}$, $|f_E| < \frac{1}{3}$ on ℓ, and $0 < f_E(6) < \frac{1}{4}$. These all hold for all sufficiently small ϵ in view of the preceding discussion. $f_E(-2) < 0$ and $f_E(2) > 0$, and so f_E has a zero ζ_1 between -2 and 2. $f_E(6) > 0$ and $f_E(9-\epsilon) = -1$ by Lemma 15.4: so f_E has a zero ζ_2 between 6 and $9-\epsilon$. ζ_1 and ζ_2 are the only finite zeros of f_E by Corollary 7.2 (a). We know that $0 \leqslant f_E < \frac{1}{3}$ on $[\zeta_1,6]$. Also $0 \leqslant f_E < \frac{1}{3}$ on $[6,\zeta_2]$: for if $f_E(x) \geqslant \frac{1}{3}$ for some $x \in]6,\zeta_2[$ then f_E must take the value $\frac{1}{4}$ at least once in each of the four intervals $]\zeta_1,2[$, $]2,6[$, $]6,x[$ and $]x,\zeta_2[$, contradicting Corollary 7.2.

Consider the set E_r, where $r = \frac{1}{3}$. E_r is symmetric, and each component of E_r contains a component of E. E_r does not meet ℓ since $|f| < \frac{1}{3}$ on ℓ. E_r must have a single component to the right of ℓ (the component containing $\overline{D}(9;\epsilon)$). $-2 \in E_r$ since $f_E(-2) < -\frac{1}{3}$. The component of E_r containing -2 must contain at least one of the components of F, and so by symmetry it contains them both. Thus E_r has precisely two components. ζ_1 and ζ_2 lie in $\Omega(E_r)$ since each of them can be joined to ∞ by means of ℓ and $[\zeta_1,\zeta_2]$, which lie in $S^2 - E_r$. Shirokov's conjecture asserts that $f_{E_r} = f_E/r$, which is impossible since f_E/r has two finite zeros in $\Omega(E_r)$, at ζ_1 and ζ_2, whereas f_{E_r} has only one by Corollary 7.2 (a).



boundary of E : continuous lines

boundary of E_r : broken lines

§16 The Domain Functions of Real Sets

Let E be a compact subset of $\underset{\sim}{R}$. In this section we shall make use of the function σ_E defined on $S^2 - E$ by:

$$\sigma_E(z) = \exp\left(\frac{1}{2}\int_E \frac{dx}{z-x}\right).$$

Clearly σ_E is analytic and symmetric, and $\sigma_E(\infty) = 1$. If $z \in \underset{\sim}{C}_L$ then $0 < \frac{1}{\pi}\,\mathrm{Im}\int_E \frac{dx}{z-x} < 1$ as we saw in §14; hence $\sigma_E(z)$ lies in the first quadrant. By symmetry, if $z \in \underset{\sim}{C}_U$ then $\sigma_E(z)$ lies in the fourth quadrant.

Suppose that E is the union of finitely many disjoint closed intervals: $E = \bigcup_{i=1}^{n}[a_i, b_i]$ where $a_i < b_i$ for each i and the intervals $[a_i, b_i]$ are disjoint. Then $\int_E \frac{dx}{z-x} = \sum_{i=1}^{n}\log\frac{z-a_i}{z-b_i}$ and so:

$$\sigma_E(z) = \prod_{i=1}^{n}\left(\frac{z-a_i}{z-b_i}\right)^{\frac{1}{2}}. \tag{1}$$

(We use that branch of the square root which takes the value 1 when $z = \infty$.)

16.1 <u>Theorem</u> Let E be a compact subset of $\underset{\sim}{R}$. Then:

$$K_E(z,\zeta) = \frac{-\sigma_E(z) + \sigma_E(\overline{\zeta})}{2\pi(z-\overline{\zeta})(4\sigma_E(z)\sigma_E(\overline{\zeta}))^{\frac{1}{2}}} \qquad (z,\zeta \in \underset{\sim}{C}-E,\ z \neq \overline{\zeta})$$

$$L_E(z,\zeta) = \frac{\sigma_E(z) + \sigma_E(\zeta)}{2\pi(z-\zeta)(4\sigma_E(z)\sigma_E(\zeta))^{\frac{1}{2}}} \qquad (z,\zeta \in \underset{\sim}{C}-E,\ z \neq \zeta)$$

where the square roots are the ones in the right half plane.

<u>Proof</u> Fix $\zeta \in \underset{\sim}{C}-E$. Denote the above expressions by $K(z)$ and $L(z)$ respectively: we have to show that $K(z) = K_E(z,\zeta)$ and that $L(z) = L_E(z,\zeta)$. By Theorem 9.4 we may assume that E is the union of finitely many disjoint closed intervals. Obviously K is analytic on S^2-E (its singularity at $\overline{\zeta}$ is removable), L is analytic on S^2-E except for a simple pole at ζ with residue $\frac{1}{2\pi}$, and K and L vanish at ∞. By (1), $\sigma_E|_{\underset{\sim}{C}_L}$ continues analytically across int E (the interior of E as a subset of $\underset{\sim}{R}$) and its values on int E lie on the upper half of the imaginary axis. Hence $K|_{\underset{\sim}{C}_L}$ and $L|_{\underset{\sim}{C}_L}$ continue analytically across int E and $\overline{K|_{\underset{\sim}{C}_L}} = -(1/i)L|_{\underset{\sim}{C}_L}$ on int E.

Throughout this proof, α_1, $\alpha_2, \ldots$ will denote functions defined in a neighbourhood of 0 which are analytic at 0 and do not vanish at 0. Let c be an endpoint of one of the intervals comprising E. By (1), $\sigma_E(z)$ has the form $(z-c)^{\frac{1}{2}}\alpha_1(z-c)$ or $(z-c)^{-\frac{1}{2}}\alpha_1(z-c)$ near c, according as c is a left-hand or a right-hand endpoint. In either case:

(2) $\quad K(z) = (z-c)^{-\frac{1}{4}}\alpha_2((z-c)^{\frac{1}{2}})$ and $L(z) = (z-c)^{-\frac{1}{4}}\alpha_3((z-c)^{\frac{1}{2}})$

near c.

By repeated application of the Riemann mapping theorem we can find a compact plane set F whose outer boundary is the union of finitely many disjoint symmetric analytic Jordan curves in $\underset{\sim}{C}$, and a symmetric conformal map ϕ of $\Omega(F)$ onto $S^2 - E$ such that $\phi(\infty) = \infty$ and $\phi'(\infty) > 0$. By the Schwarz reflection principle, ϕ continues analytically across $\partial\Omega(F)$. Write $\zeta' = \phi^{-1}(\zeta)$. Define functions K_1 and L_1 on $\Omega(F)$ by:

$$(3) \qquad \begin{aligned} K_1(z) &= \phi'(z)^{\frac{1}{2}}\, \overline{\phi'(\zeta')^{\frac{1}{2}}}\, K(\phi(z)) \\ L_1(z) &= \phi'(z)^{\frac{1}{2}}\, \phi'(\zeta')^{\frac{1}{2}}\, L(\phi(z)). \end{aligned}$$

Clearly, K_1 is analytic on $\Omega(F)$, L_1 is analytic on $\Omega(F)$ except for a simple pole at ζ' with residue $\frac{1}{2\pi}$, and K_1 and L_1 vanish at ∞. On $(\partial\Omega(F))_L$, K_1 and L_1 are analytic and $\overline{K_1(z)}ds = \overline{\phi'(z)^{\frac{1}{2}}}\, \phi'(\zeta')^{\frac{1}{2}}\, \overline{K|_{\underset{\sim}{C}_L}(\phi(z))}ds = -\frac{1}{i}\phi'(z)^{\frac{1}{2}}\, \phi'(\zeta)^{\frac{1}{2}}\, L|_{\underset{\sim}{C}_L}(\phi(z))dz = -\frac{1}{i}L_1(z)dz$. Similarly $\overline{K_1(z)}ds = -\frac{1}{i}L_1(z)dz$ on $(\partial\Omega(F))_U$.

We claim that K_1 and L_1 are also analytic at each of the points where $\partial\Omega(F)$ meets $\underset{\sim}{R}$, and hence are analytic on the whole of $\partial\Omega(F)$. Let x_0 be any of the points where $\partial\Omega(F)$ meets $\underset{\sim}{R}$. Write $c = \phi(x_0)$. Then c is an endpoint of one of the intervals comprising E. It is clear from the behaviour of ϕ near x_0 that $\phi'(x_0) = 0$ and $\phi''(x_0) \neq 0$. Hence $\phi(z) = c + (z-x_0)^2\alpha_4(z-x_0)$ and $\phi'(z) = (z-x_0)\alpha_5(z-x_0)$ near x_0. Therefore, near x_0:

$$\begin{aligned} K_1(z) &= \phi'(z)^{\frac{1}{2}}\, \overline{\phi'(\zeta')^{\frac{1}{2}}}\, K(\phi(z)) \\ &= \phi'(z)^{\frac{1}{2}}\, \overline{\phi'(\zeta')^{\frac{1}{2}}}\, (\phi(z)-c)^{-\frac{1}{4}}\, \alpha_2((\phi(z)-c)^{\frac{1}{2}}) \qquad \text{by } (2) \\ &= \{(z-x_0)\alpha_5(z-x_0)\}^{\frac{1}{2}}\, \overline{\phi'(\zeta')^{\frac{1}{2}}}\, \{(z-x_0)^2\alpha_4(z-x_0)\}^{-\frac{1}{4}}\, \alpha_2\{(z-x_0)(\alpha_4(z-x_0))^{\frac{1}{2}}\} \\ &= (z-x_0)^{\frac{1}{2}}\, (z-x_0)^{-\frac{1}{2}}\, \alpha_6(z-x_0) \\ &= \alpha_6(z-x_0). \end{aligned}$$

So K_1 is analytic at x_0. Similarly, L_1 is analytic at x_0.

We have shown that K_1 is analytic on $\overline{\Omega(F)}$, that L_1 is analytic on $\overline{\Omega(F)}$ except for a simple pole at ζ' with residue $\frac{1}{2\pi}$, that K_1 and L_1 vanish at ∞, and that $\overline{K_1(z)}ds = -\frac{1}{i}L_1(z)dz$ round $\partial\Omega(F)$. Hence by the discussion in §8, K_1 is the Szegö kernel and L_1 the co-kernel of $\Omega(F)$ at ζ'. By (3) and Theorem 9.5, K is the Szegö kernel and L the co-kernel of $S^2 - E$ at ζ.

16.2 <u>Corollary</u> Let E be a compact subset of $\underset{\sim}{R}$ and let $\gamma(E) > 0$. Let $z \in S^2 - E$. Then:

$$\gamma(E)K_E^+(z,\infty) = \frac{\sigma_E(z) + 1}{2\pi \; 2\sigma_E(z)^{\frac{1}{2}}}$$

$$\gamma(E)K_E^+(z,\infty)f_E(z) = \frac{\sigma_E(z) - 1}{2\pi \; 2\sigma_E(z)^{\frac{1}{2}}}.$$

<u>Proof</u> The results follow from Theorem 16.1 and Proposition 10.2.

Notice that dividing the second formula in Corollary 16.2 by the first gives $f_E(z) = \dfrac{\sigma_E(z) - 1}{\sigma_E(z) + 1}$, as we saw in §14.

16.3 <u>Corollary</u> Let E be a compact subset of $\underset{\sim}{R}$. Let $z \in \underset{\sim}{C} - E$. Then $a_E(z) = \cos\beta$, where β is half of the angle subtended by E at z.

<u>Proof</u> Abbreviate $\sigma_E(z)$ to σ. Since $\sigma = \exp\left(\frac{1}{2}\int_E \frac{dx}{z-x}\right)$, $|\arg\sigma| = \frac{1}{2}\left|\operatorname{Im}\int_E \frac{dx}{z-x}\right| = \frac{1}{2}\int_E \frac{|\operatorname{Im} z|\,dx}{(\operatorname{Re} z - x)^2 + (\operatorname{Im} z)^2} = \beta$. $f_E(z) = \dfrac{\sigma - 1}{\sigma + 1}$, and $\psi_E(z) = \frac{2\pi}{i}\gamma(E)^2K_E^+(z,\infty)^2 = \frac{1}{2\pi i}\frac{(\sigma+1)^2}{4\sigma}$. So $a_E(z) =$ $2\pi|\psi_E(z)|\{1 - |f_E(z)|^2\} = \frac{1}{4|\sigma|}(|\sigma+1|^2 - |\sigma-1|^2) = \frac{\operatorname{Re}\sigma}{|\sigma|} =$ $\cos(|\arg\sigma|) = \cos\beta$.

Corollary 16.3 shows that, for compact subsets of $\underset{\sim}{R}$ at least, a_E is a decreasing function of E and is at most 1.

If E is a compact subset of $\bar{\underset{\sim}{R}}$ and $\infty \in \operatorname{int} E$, then we can again define $\sigma_E(z) = \exp\left(\frac{1}{2}\int_E \frac{dx}{z-x}\right)$. The integral must be interpreted in the Cauchy sense, i.e. $\lim\limits_{R\to\infty}\int_{E\cup[-R,R]} dx/(z-x)$. Theorem 16.1 still holds in this case, and the proof is almost unchanged.

Now that we have explicit formulae for K_E and L_E, where E is a compact subset of $\underset{\sim}{R}$, we can make a direct attack on the problem of finding the sign of $b_E(z,\zeta)$. One tool we shall need is a slightly more general form of Schwarz's lemma. Let V be a simply-connected domain in S^2. We define the <u>Gleason metric</u> d_V on V as follows. Let $z, \zeta \in V$, and let ϕ be a conformal mapping of V onto D such that $\phi(\zeta) = 0$. Then we write $d_V(z,\zeta) = |\phi(z)|$. This is well defined since ϕ is unique up to a rotation. It is easy to verify that d_V is a metric on V, though we shall not use that fact.

16.4 <u>Theorem</u> Let U and V be simply-connected domains in S^2, let $z, \zeta \in U$, and let ϕ be an analytic function of U into V. Then $d_V(\phi(z),\phi(\zeta)) \leqslant d_U(z,\zeta)$.

<u>Proof</u> This follows from the case when $U = V = D$, $\zeta = \phi(\zeta) = 0$, which is well-known.

If $\zeta \in \underset{\sim}{C}_L$, then the mapping $z \to \dfrac{z-\zeta}{z-\bar{\zeta}}$ maps $\underset{\sim}{C}_L$ conformally onto D and takes ζ to 0. So the Gleason metric for $\underset{\sim}{C}_L$ is $d_{\underset{\sim}{C}_L}(z,\zeta) = \dfrac{|z-\zeta|}{|z-\bar{\zeta}|}$. Hence also the Gleason metric for the first quadrant is $d(z,\zeta) = \dfrac{|z^2-\zeta^2|}{|z^2-\bar{\zeta}^2|}$. Observe that these expressions are jointly continuous in z and ζ.

16.5 <u>Lemma</u> Let $0 < r < R$. For each $t \in]-r,r[$ let A_t and B_t be the points in $\underset{\sim}{C}_U$ at which the line $\{z : \mathrm{Re}\, z = t\}$ meets the circles $\{z : |z| = r\}$ and $\{z : |z| = R\}$ respectively. Then the acute angle θ_t between $0A_t$ and $0B_t$ increases with $|t|$.

<u>Proof</u> We may restrict attention to positive t.

$$\theta_t = \cos^{-1}\frac{t}{R} - \cos^{-1}\frac{t}{r}$$

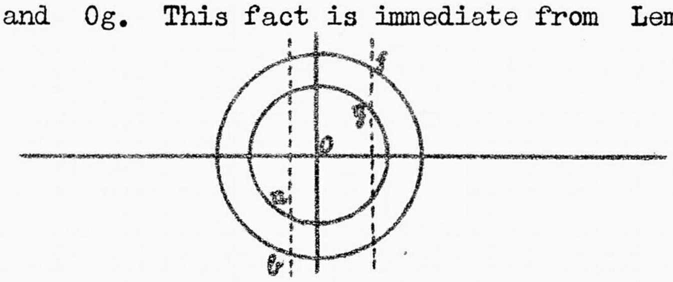
$$\frac{d\theta_t}{dt} = \frac{1}{\sqrt{r^2 - t^2}} - \frac{1}{\sqrt{R^2 - t^2}} > 0.$$

16.6 <u>Lemma</u> Let a, b, f and g be complex numbers such that $\mathrm{Re}\, a = \mathrm{Re}\, b$ and $\mathrm{Re}\, f = \mathrm{Re}\, g$. Then:

(a) $\mathrm{Im}\, b \leq \mathrm{Im}\, a < 0$ & $\mathrm{Im}\, f > 0$ & $\left|\frac{f}{b}\right| \geq \left|\frac{g}{a}\right| \Rightarrow \mathrm{Re}\,\frac{f}{b} - \mathrm{Re}\,\frac{g}{a} \leq 0$;

(b) $\mathrm{Im}\, a \leq \mathrm{Im}\, b < 0$ & $\mathrm{Im}\, g < 0$ & $\left|\frac{f}{b}\right| \leq \left|\frac{g}{a}\right| \Rightarrow \mathrm{Re}\,\frac{f}{b} - \mathrm{Re}\,\frac{g}{a} \leq 0$.

<u>Proof</u> We shall prove (a); to obtain (b) from (a), replace a by b, b by a, f by $-g$, and g by $-f$. Since f and g can be multiplied by a positive constant without affecting the hypotheses or the conclusion, we may assume that $\left|\frac{f}{b}\right| = 1$. Fix a, b, and f. $|g| \leq |a|$; so g lies on that segment of the vertical line through f cut off by the circle centred at 0 which passes through a. In fact we can assume that g is the upper endpoint of that segment, since the higher g is, the greater is $-\mathrm{Re}\,\frac{g}{a}$. Now $\left|\frac{f}{b}\right| = \left|\frac{g}{a}\right| = 1$; and so the problem reduces to showing that the smaller angle between $0b$ and $0f$ is at least as great as the smaller angle between $0a$ and $0g$. This fact is immediate from Lemma 16.5.



16.7 Theorem Let $E \in J$. Let $z, \zeta \in \Omega(E)$, $z \neq \zeta$. Then $b_E(z,\zeta) \leq 0$.

Proof Since the sign of b_E is invariant under a conformal mapping which maps ∞ to ∞ (see §13), we may assume that E is a compact subset of $\underset{\sim}{R}$. Let z and ζ be distinct finite points of $\Omega(E)$. It is sufficient to prove the result in the case when $z \neq \bar{\zeta}$, as the case $z = \bar{\zeta}$ then follows by continuity of b_E. For this proof only, we shall use the temporary abbreviations $p = \sigma_E(z)$ and $q = \sigma_E(\zeta)$. Substituting the values for the domain functions of $S^2 - E$ given by Theorem 16.1 and Corollary 16.2 into the formula (2) of §13 gives:

$$b_E(z,\zeta) = 2\,\mathrm{Re}\left(\frac{-p+\bar{q}}{(z-\bar{\zeta})(4p\bar{q})^{\frac{1}{2}}}\,\frac{q-1}{2q^{\frac{1}{2}}}\,\frac{\bar{p}-1}{2\bar{p}^{\frac{1}{2}}} - \frac{-p+\bar{q}}{(z-\bar{\zeta})(4p\bar{q})^{\frac{1}{2}}}\,\frac{q+1}{2q^{\frac{1}{2}}}\,\frac{\bar{p}+1}{2\bar{p}^{\frac{1}{2}}}\right.$$

$$\left. + \frac{p+q}{(z-\zeta)(4pq)^{\frac{1}{2}}}\,\frac{\bar{q}-1}{2\bar{q}^{\frac{1}{2}}}\,\frac{\bar{p}+1}{2\bar{p}^{\frac{1}{2}}} - \frac{p+q}{(z-\zeta)(4pq)^{\frac{1}{2}}}\,\frac{\bar{q}+1}{2\bar{q}^{\frac{1}{2}}}\,\frac{\bar{p}-1}{2\bar{p}^{\frac{1}{2}}}\right)$$

$$= \frac{1}{2|p|\,|q|}\left(\mathrm{Re}\,\frac{(p-\bar{q})(\bar{p}+q)}{z-\bar{\zeta}} - \mathrm{Re}\,\frac{(p+q)(\bar{p}-\bar{q})}{z-\zeta}\right).$$

Write $a = z-\zeta$, $b = z-\bar{\zeta}$, $f = (p-\bar{q})(\bar{p}+q)$, and $g = (p+q)(\bar{p}-\bar{q})$. We have to show that $\mathrm{Re}\,\frac{f}{b} - \mathrm{Re}\,\frac{g}{a} \leq 0$. We distinguish three cases.

Case 1: z and ζ lie on the same side of the real axis. We may assume that $\mathrm{Im}\, z \leq \mathrm{Im}\,\zeta < 0$. Clearly $\mathrm{Re}\, a = \mathrm{Re}\, b$ ($= \mathrm{Re}\, z - \mathrm{Re}\,\zeta$), $\mathrm{Re}\, f = \mathrm{Re}\, g$ ($= |p|^2 - |q|^2$), $\mathrm{Im}\, b \leq \mathrm{Im}\, a < 0$, and $\mathrm{Im}\, f = 2\,\mathrm{Im}\, pq > 0$ since p and q lie in the first quadrant. By Lemma 16.6 (a), all that remains to be shown is that $\left|\frac{f}{b}\right| \geq \left|\frac{g}{a}\right|$. Now σ_E maps $\underset{\sim}{C}_L$ into the first quadrant: and applying Schwarz's lemma (Theorem 16.4) to σ_E at the points z and ζ we find that:

$$\frac{|p^2 - q^2|}{|p^2 - \bar{q}^2|} \leq \frac{|z-\zeta|}{|z-\bar{\zeta}|},$$

i.e. that:

$$\frac{|p-\bar{q}|\,|\bar{p}+q|}{|z-\bar{\zeta}|} \geqslant \frac{|p+q|\,|\bar{p}-\bar{q}|}{|z-\zeta|}$$

as required.

Case 2: z and ζ lie on opposite sides of the real axis. We may assume that $\operatorname{Im} z \leqslant -\operatorname{Im} \zeta < 0$. Applying Schwarz's lemma to σ_E at the points z and $\bar{\zeta}$ we find that $\dfrac{|p^2-\bar{q}^2|}{|p^2-q^2|} \leqslant \dfrac{|z-\bar{\zeta}|}{|z-\zeta|}$, i.e. that $\left|\dfrac{f}{b}\right| \leqslant \left|\dfrac{g}{a}\right|$. Lemma 16.6 (b), whose other hypotheses are easily verified, gives the result.

Case 3: z or ζ lies on the real axis. The result follows from Case 1 by continuity of b_E.

16.8 Corollary Let E be a compact subset of S^2 with at most two components. Let z, ζ, and a be distinct points of $\underset{\sim}{C}-E$. Then $b_E(z,\zeta,a) \leqslant 0$.

Proof Apply a linear fractional transformation which takes a to ∞, and then use Proposition 15.2 and Theorem 16.7.

If E is a compact subset of $\underset{\sim}{R}$ and z, ζ, and a are distinct points of $\underset{\sim}{C}-E$ then Theorem 16.1 and formula (1) of §13 give an explicit expression for $b_E(z,\zeta,a)$. I do not know whether it is always non-positive: but there is an interesting attempt at a proof, which we shall now outline. We shall consider the case when z, ζ, and a all lie in the lower half plane: the other cases are dealt with similarly. Theorem 16.1 and formula (1) of §13 show that $b_E(z,\zeta,a) = G(\sigma_E(z),\sigma_E(\zeta),\sigma_E(a))$, where G is the function given by:

$$G(p,q,r) = \frac{1}{4|p|\,|q|\,|r|}\operatorname{Re}\left(\frac{(-p+\bar{q})(-q+\bar{r})(-r+\bar{p})}{(z-\bar{\zeta})(\zeta-\bar{a})(a-\bar{z})} + \frac{(-p+\bar{q})(q+r)(\bar{r}+\bar{p})}{(z-\bar{\zeta})(\zeta-a)(\bar{a}-\bar{z})} + \frac{(\bar{p}+\bar{q})(-q+\bar{r})(r+p)}{(\bar{z}-\bar{\zeta})(\zeta-\bar{a})(a-z)} + \frac{(p+q)(\bar{q}+\bar{r})(-r+\bar{p})}{(z-\zeta)(\bar{\zeta}-\bar{a})(a-\bar{z})}\right).$$

Denote the first quadrant by Q. Write $R = \{(p,q,r) \in \underset{\sim}{C}^3 : p = \sigma(z)$, $q = \sigma(\zeta)$ and $r = \sigma(a)$ for some analytic function $\sigma : \underset{\sim}{C}_L \to Q\}$. It would be sufficient to show that $G \leq 0$ on R. R is a relatively closed subset of Q^3. R is convex (since Q is convex). R can be described explicitly, using the Gleason metric. It can be shown, either by an "extremal problem" technique similar to the one in §7 or by means of the explicit description of R, that if $(p_0, q_0, r_0) \in \partial R$ and $p_0 = \sigma_0(z)$, $q_0 = \sigma_0(\zeta)$ and $r_0 = \sigma_0(a)$ for some analytic function $\sigma_0 : \underset{\sim}{C}_L \to Q$, then σ_0 extends to a continuous mapping of $\overline{\underset{\sim}{C}_L}$ onto $\overline{Q}$, σ_0 maps $\partial\underset{\sim}{C}_L$ onto ∂Q, and as z runs round $\partial\underset{\sim}{C}_L$, $\sigma_0(z)$ runs either once or twice round ∂Q. We shall show that $G(\sigma(z), \sigma(\zeta), \sigma(a)) \leq 0$ for every such mapping σ. Let σ be such a mapping. We can assume that $\sigma(\infty)$ is neither 0 nor ∞: an easy continuity argument takes care of these cases. Write $E = \{x \in \overline{\underset{\sim}{R}} : \sigma(x) \in \overline{i\underset{\sim}{R}}\}$. E is either a closed interval in $\overline{\underset{\sim}{R}}$ or the union of two disjoint closed intervals in $\overline{\underset{\sim}{R}}$, and ∞ either is not in E or is an interior point of E. σ and σ_E both map $\underset{\sim}{C}_L$ into Q and extend continuously to $\overline{\underset{\sim}{R}}$, taking real values on $\overline{\underset{\sim}{R}} - E$ and purely imaginary values on $\operatorname{int} E$. $\frac{\sigma}{\sigma_E}$ is an analytic function of $\underset{\sim}{C}_L$ into the right half plane, and extends continuously to, and takes real values at, every point of $\overline{\underset{\sim}{R}}$ except possibly the endpoints of the intervals comprising E; so $\frac{\sigma}{\sigma_E}$ is a positive constant. Hence $G(\sigma(z), \sigma(\zeta), \sigma(a)) = G(\sigma_E(z), \sigma_E(\zeta), \sigma_E(a)) = b_E(z, \zeta, a) \leq 0$ by Corollary 16.8. This shows that $G \leq 0$ on $Q^3 \cap \partial R$. However, I can find no way of extending the result to the whole of R.

If Ω is any domain in S^2 whose complement has three components, none of them a singleton, then Ω can be mapped conformally onto the complement in S^2 of the union of three disjoint closed intervals in $\underset{\sim}{R}$. (The proof is similar to the proof of Proposition 15.2.) Hence if we could show that $b_E(z,\zeta,a) \leqslant 0$ whenever E is a compact subset of $\underset{\sim}{R}$ then it would follow that $b_E \leqslant 0$ whenever E is a compact subset of S^2 with at most three components.

CHAPTER V

CURVE EXTENSION

In §13 we saw, at a superficial level, the relationship between the subadditivity problem and the perturbation problems of §11 and §12. That relationship is the theme of this final chapter.

§17 A Second Perturbation Problem

In Chapter III we examined the effect of removing a small disc from a domain Ω. There are, of course, other ways of perturbing Ω. If Ω is bounded by finitely many smooth Jordan curves then we could study small displacements of these curves: this idea gives rise to a very elegant method, the Hadamard variational method, in the study of the Bergman kernel (see [3], chapter 8), but does not make a good job of the Szegö kernel. In this section we look at the following type of perturbation: we start with a compact plane set E consisting of the union of an analytic arc Γ with one endpoint a and a compact plane set not containing a, we assume that $a \in \partial\Omega(E)$, and we perturb $\Omega(E)$ by continuing Γ beyond a. Theorem 17.14 below gives reason to believe that this form of perturbation is a natural one to look at. But unfortunately it is very awkward to handle, and consequently the results in this section are rather patchy, the proofs have an ad-hoc character, and the general theory which one might confidently predict is just too hard. Nevertheless, even the results which can be salvaged will yield some interesting subadditivity theorems in the next section.

We shall want to know how certain types of curve transform under the mappings $z \to z^2$ and $z \to z^{\frac{1}{2}}$. The easiest approach is to look at such curves in polar co-ordinates.

17.1 Lemma (a) An arc which, near 0, has a cartesian equation of the form:

$$y = \alpha_2 x^2 + \alpha_3 x^3 + \ldots\ldots \qquad (x \geq 0)$$

has, near 0, a polar equation of the form:

$$\theta = \beta_1 r + \beta_2 r^2 + \ldots\ldots \qquad (r \geq 0)$$

and vice versa.

(b) An arc which, near 0, has a cartesian equation of the form:

$$y = \alpha_3 x^3 + \alpha_5 x^5 + \alpha_7 x^7 + \ldots\ldots \qquad (x \geq 0)$$

has, near 0, a polar equation of the form:

$$\theta = \beta_2 r^3 + \beta_4 r^4 + \beta_6 r^6 + \ldots\ldots \qquad (r \geq 0)$$

and vice versa.

(c) An arc which, near 0, has a cartesian equation of the form:

$$y = \alpha_{\frac{3}{2}} x^{\frac{3}{2}} + \alpha_2 x^2 + \alpha_{\frac{5}{2}} x^{\frac{5}{2}} + \ldots\ldots \qquad (x \geq 0)$$

has, near 0, a polar equation of the form:

$$\theta = \beta_{\frac{1}{2}} r^{\frac{1}{2}} + \beta_1 r + \beta_{\frac{3}{2}} r^{\frac{3}{2}} + \ldots\ldots \qquad (r \geq 0)$$

and vice versa.

Proof The proofs are absolutely mechanical. We shall prove (a) in one direction as an example. α_i, β_i etc. denote real constants.

$$y = \alpha_2 x^2 + \alpha_3 x^3 + \ldots\ldots \qquad (x \geqslant 0)$$

$$r^2 = x^2 + y^2 = x^2 + \gamma_4 x^4 + \gamma_5 x^5 + \ldots\ldots$$

$$= x^2(1 + \gamma_4 x^2 + \gamma_5 x^3 + \ldots\ldots)$$

$$(1) \qquad r = x + \delta_3 x^3 + \delta_4 x^4 + \ldots\ldots$$

$$\tan\,\theta = \frac{y}{x} = \alpha_2 x + \alpha_3 x^2 + \ldots\ldots$$

$$\theta = \tan^{-1}(\alpha_2 x + \alpha_3 x^2 + \ldots\ldots)$$

$$= \epsilon_1 x + \epsilon_2 x^2 + \ldots\ldots$$

$$= \beta_1 r + \beta_2 r^2 + \ldots\ldots \qquad \text{by (1).}$$

The arcs in Lemma 17.1 (a) are clearly just analytic arcs entering 0 along the positive real axis. Arcs of the types considered in (a), (b) and (c) will be called, respectively, <u>analytic arcs starting at</u> 0 <u>in the direction</u> 1, <u>symmetric analytic arcs starting at</u> 0 <u>in the direction</u> 1, and <u>$\frac{1}{2}$- arcs starting at</u> 0 <u>in the direction</u> 1. If Γ is an analytic arc (respectively, a symmetric analytic arc or a $\frac{1}{2}$- arc) starting at 0 in the direction 1, and if $a \in \underset{\sim}{C}$, $\lambda \in \underset{\sim}{C}$ and $|\lambda| = 1$, then $\lambda\Gamma + a$ will be called an <u>analytic arc</u> (respectively, a <u>symmetric analytic arc</u> or a <u>$\frac{1}{2}$- arc</u>) <u>starting at</u> a <u>in the direction</u> λ.

The polar forms show immediately that, under the mapping $z \to z^{\frac{1}{2}}$, near 0, the symmetric analytic arcs starting at 0 in the direction 1 are precisely the images of the analytic arcs starting at 0 in the direction 1, and these in turn are precisely the images of the $\frac{1}{2}$- arcs starting at 0 in the direction 1.

If Γ is a symmetric analytic arc starting at 0, then the cartesian form of Γ shows that $\Gamma \cup (-\Gamma)$ is analytic at 0. (This explains the name.)

Suppose that $\Omega \in G$, where Ω is one of the complementary components of a compact subset E of S^2. Suppose that $0 \in \partial\Omega$. If ζ_1 and ζ_2 are finite points of Ω then by Theorem 12.1 there is a non-zero complex number β such that $\mathrm{Pert}(K(\zeta_1,\zeta_2),z) = \beta|z| + o(|z|)$ as $z \to 0$ along any curve normal to $\partial\Omega$ at 0. If in addition $\zeta_1 \neq \zeta_2$ then the same result holds with K replaced by L. The following lemma shows that this is not typical of the behaviour of the sets in which we are now interested.

17.2 Lemma Let E be a compact subset of $\underset{\sim}{C}$. Suppose that E has finitely many components, and is the union of an analytic arc starting at 0 in the direction -1 and a compact set not containing 0. Suppose that $0 \in \partial\Omega(E)$. Let ζ_1 and ζ_2 be finite points of $\Omega(E)$. Then there is a non-zero complex number α such that, whenever $z \to 0$ along any curve entering 0 along the positive real axis, $\mathrm{Pert}(K_E(\zeta_1,\zeta_2);z) \to \alpha$. If in addition $\zeta_1 \neq \zeta_2$ then the same result holds with K replaced by L.

Proof Map $\Omega(E)$ conformally by some map ϕ onto a domain $\Omega(E') \in G$ so that the component of E' which corresponds to the component of E containing 0 is the closed disc with centre $-r$ and radius r for some $r > 0$, and so that $\phi(0) = 0$. The mapping $z \to z^{\frac{1}{2}}$ defined off E near 0 maps the boundary of E near 0, by Lemma 17.1, onto a (symmetric) analytic arc whose tangent at 0 is vertical. It follows that, near 0, $\phi(z) = \psi(z^{\frac{1}{2}})$, where ψ is analytic at 0, $\psi(0) = 0$, and $\psi'(0) > 0$. We can choose r so that $\psi'(0) = 1$. Now, if z is on any curve entering 0 along the positive real axis, then by Theorem 12.1 and Theorem 9.5:

$$\mathrm{Pert}(K_E(\zeta_1,\zeta_2);z) = |\phi'(z)|\phi'(\zeta_1)^{\frac{1}{2}}\overline{\phi'(\zeta_2)^{\frac{1}{2}}}\,\mathrm{Pert}\{K_{E'}(\phi(\zeta_1),\phi(\zeta_2));\phi(z)\}$$

$$= \tfrac{1}{2}|z|^{-\frac{1}{2}}(1 + O(|z|^{\frac{1}{2}}))\phi'(\zeta_1)^{\frac{1}{2}}\overline{\phi'(\zeta_2)^{\frac{1}{2}}}(\beta|z|^{\frac{1}{2}} + o(|z|^{\frac{1}{2}}))$$

since $\phi(z) = \psi(z^{\frac{1}{2}})$; here $\beta \neq 0$. Hence $\mathrm{Pert}(K_E(\zeta_1,\zeta_2);z) \to \frac{1}{2}\beta\phi'(\zeta_1)^{\frac{1}{2}}\overline{\phi'(\zeta_2)^{\frac{1}{2}}} \neq 0$. The proof for L is similar.

17.3 <u>Lemma</u> Let E be a compact subset of $\underset{\sim}{R}$. Suppose that E contains all negative numbers sufficiently near 0 and no positive numbers sufficiently near 0. Let $\zeta_1, \zeta_2 \in \underset{\sim}{C} - E$. Then for $x > 0$:

$$K_{E\cup[0,x]}(\zeta_1,\zeta_2) = K_E(\zeta_1,\zeta_2) + \alpha x + O(x^2)$$

where:

$$\alpha = \tfrac{1}{4}\lim_{x\to 0+} \mathrm{Pert}(K_E(\zeta_1,\zeta_2);x).$$

If in addition $\zeta_1 \neq \zeta_2$ then the same result holds with K replaced throughout by L.

<u>Proof</u> The proof is a matter of direct computation: we shall prove the result for K. We shall assume that $\zeta_1 \neq \overline{\zeta_2}$: the case when $\zeta_1 = \overline{\zeta_2}$ needs a separate (though equally automatic) proof. Write:

$$p(x) = \exp\left(\tfrac{1}{2}\int_{E\cup[0,x]}\frac{ds}{\zeta_1 - s}\right), \qquad p = p(0), \qquad p' = p'(0) = \frac{p}{2\zeta_1},$$

$$q(x) = \exp\left(\tfrac{1}{2}\int_{E\cup[0,x]}\frac{ds}{\zeta_2 - s}\right), \qquad q = q(0), \qquad q' = q'(0) = \frac{q}{2\zeta_2},$$

$$r = \exp\left(\tfrac{1}{2}\int_E \frac{ds}{x - s}\right).$$

Now $p(x) = p + xp' + O(x^2)$ and $q(x) = q + xq' + O(x^2)$. Hence by Theorem 16.1:

$$K_{E \cup [0,x]}(\zeta_1,\zeta_2) = \frac{-p(x) + \overline{q(x)}}{2\pi(\zeta_1 - \overline{\zeta_2})(4p(x)\overline{q(x)})^{\frac{1}{2}}} = K_E(\zeta_1,\zeta_2) + \alpha x + O(x^2)$$

where:

$$\alpha = \frac{d^+}{dx} K_{E \cup [0,x]}(\zeta_1,\zeta_2)\Big|_{x=0} = \frac{(4p\overline{q})^{\frac{1}{2}}(-p'+\overline{q'}) + (p-\overline{q})\frac{1}{2}(4p\overline{q})^{-\frac{1}{2}}(4p\overline{q'} + 4p'\overline{q})}{2\pi(\zeta_1 - \overline{\zeta_2})4p\overline{q}}$$

$$= \frac{1}{2\pi(\zeta_1 - \overline{\zeta_2})(4p\overline{q})^{\frac{1}{2}}}\left(-\frac{p}{4\zeta_1} + \frac{\overline{q}}{4\overline{\zeta_2}} + \frac{p}{4\overline{\zeta_2}} - \frac{\overline{q}}{4\zeta_1}\right) \quad \text{since} \quad p' = \frac{p}{2\zeta_1},\ q' = \frac{q}{2\zeta_2}$$

$$= \frac{1}{4}\,\frac{p+\overline{q}}{2\pi\zeta_1\overline{\zeta_2}(4p\overline{q})^{\frac{1}{2}}}\,.$$

($\frac{d^+}{dx}$ denotes the right-hand derivative.) Also by Theorem 12.1 and Theorem 16.1, since $r, x \in \underset{\sim}{R}$:

$$\mathrm{Pert}(K_E(\zeta_1,\zeta_2);x) = 2\pi\{L_E(x,\zeta_1)\overline{L_E(x,\zeta_2)} - \overline{K_E(x,\zeta_1)}K_E(x,\zeta_2)\}$$

$$= \frac{r+p}{2\pi(x-\zeta_1)(4rp)^{\frac{1}{2}}}\,\frac{r+\overline{q}}{2\pi(x-\overline{\zeta_2})(4r\overline{q})^{\frac{1}{2}}} - \frac{-r+p}{2\pi(x-\zeta_1)(4rp)^{\frac{1}{2}}}\,\frac{-r+\overline{q}}{2\pi(x-\overline{\zeta_2})(4r\overline{q})^{\frac{1}{2}}}$$

$$= \frac{p+\overline{q}}{2\pi(x-\zeta_1)(x-\overline{\zeta_2})(4p\overline{q})^{\frac{1}{2}}}$$

$$\to \frac{p+\overline{q}}{2\pi\zeta_1\overline{\zeta_2}(4p\overline{q})^{\frac{1}{2}}} \quad \text{as} \quad x \to 0+$$

$$= 4\alpha.$$

The next lemma gives a similar result for a different type of set. Its proof consists of applying a conformal mapping to the result of Lemma 17.3.

17.4 <u>Lemma</u> Let E be a compact subset of $\underset{\sim}{C}$, and consist either of one component, not a singleton, or of two separated components, neither one a singleton. Let $a \in E$, $\lambda \in \underset{\sim}{C}$, $|\lambda| = 1$. Suppose that E is the union of an analytic arc Γ starting at a in the direction $-\lambda$ and a compact set not containing a. Let ζ_1 and ζ_2 be finite points of $\Omega(E)$. Let α be a quarter of the limit of $\mathrm{Pert}(K_E(\zeta_1,\zeta_2);z)$ as $z \to a$ along any curve entering a along the line $\{a + t\lambda : t \geq 0\}$. Then there is a $\frac{1}{2}$-arc C, starting at a in the direction λ, such that:

$$K_{E_t}(\zeta_1,\zeta_2) = K_E(\zeta_1,\zeta_2) + \alpha t + O(t^2)$$

where E_t is the union of E and a length t of C starting at a. If in addition $\zeta_1 \neq \zeta_2$ then the same result holds with K replaced throughout by L.

<u>Proof</u> We may suppose that $a \in \partial\Omega(E)$, as otherwise $\alpha = 0$ and the result is trivial. Take $a = 0$ and $\lambda = 1$ without loss. α exists by Lemma 17.2. There is a conformal map ϕ of $\Omega(E)$, which extends to be continuous at 0, which maps 0 to 0, which does not map ζ_1 or ζ_2 to ∞, and which maps $\Omega(E)$ onto $S^2 - E'$, where E' is a set of the form $[x_1,x_2] \cup [x_3,0]$ $(x_1 < x_2 \leq x_3 < 0)$. (If $\Omega(E)$ is simply-connected then that is easy. If $\Omega(E)$ is doubly-connected map $\Omega(E)$ conformally onto an annulus $\{z : 1 < |z| < R\}$ so that 0 is mapped to -1, invert the annulus about any point $\zeta \in]-R,-1[$, apply Proposition 14.3, and translate by a real constant. The leeway in the choice of ζ can be used to avoid the co-incidence of $\phi(\zeta_1)$ or $\phi(\zeta_2)$ with ∞. We do not, indeed we could not, insist that $\phi(\infty) = \infty$.) Now the mapping $z \to z^{\frac{1}{2}}$, near 0, maps E onto a symmetric analytic curve whose tangent at 0 is vertical, and it maps E' into the imaginary axis. Hence, near 0, $\phi(z) = \psi(z^{\frac{1}{2}})^2$, where ψ is analytic at 0, $\psi(0) = 0$, $\psi'(0) > 0$, and $\psi''(0)$ is purely imaginary. By replacing E' by kE' for some $k > 0$, we can assume that $\psi'(0) = 1$. Now let $C = \phi^{-1}[0,1]$. By Lemma 17.1, C is a $\frac{1}{2}$-arc starting at 0 in the direction 1. Near 0, $\phi(z) = z + \beta_2 z^2 + \ldots\ldots$, where β_2 is purely imaginary. Hence:

$$\begin{aligned} \phi(z) &= \psi(z^{\frac{1}{2}})^2 \\ &= z + 2\beta_2 z^{\frac{3}{2}} + \ldots\ldots \\ \phi'(z) &= 1 + 3\beta_2 z^{\frac{1}{2}} + 0(|z|). \end{aligned}$$

Since β_2 is purely imaginary, $|\phi'(z)| = 1 + 0(|z|)$ on C. It follows by an easy calculation that for small t the image of E_t under ϕ is $E' \cup [0,s(t)]$ where $s(t) = t + 0(t^2)$. So by Theorem 9.5:

$$\begin{aligned} K_{E_t}(\zeta_1,\zeta_2) &= \phi'(\zeta_1)^{\frac{1}{2}}\overline{\phi'(\zeta_2)^{\frac{1}{2}}} K_{E'\cup[0,s(t)]}(\phi(\zeta_1),\phi(\zeta_2)) \\ &= \phi'(\zeta_1)^{\frac{1}{2}}\overline{\phi'(\zeta_2)^{\frac{1}{2}}}\{K_{E'}(\phi(\zeta_1),\phi(\zeta_2)) + \alpha' s(t) + 0(t^2)\} \end{aligned}$$

by Lemma 17.3, where $\alpha' = \frac{1}{4}\lim \operatorname{Pert}(K_{E'}(\phi(\zeta_1),\phi(\zeta_2));z)$, the limit being taken along any curve entering 0 along the positive real axis. But $\phi'(\zeta_1)^{\frac{1}{2}}\overline{\phi'(\zeta_2)^{\frac{1}{2}}} K_{E'}(\phi(\zeta_1),\phi(\zeta_2)) = K_E(\zeta_1,\zeta_2)$, $s(t) = t + 0(t^2)$, and $\phi'(\zeta_1)^{\frac{1}{2}}\overline{\phi'(\zeta_2)^{\frac{1}{2}}}\alpha' = \alpha$ since $|\phi'(z)| \to 1$ as $z \to 0$. So $K_{E_t}(\zeta_1,\zeta_2) = K_E(\zeta_1,\zeta_2) + \alpha t + 0(t^2)$ as required. The proof for L is similar.

The rest of this section is devoted to improving Lemma 17.4 by showing that the result holds for every $\frac{1}{2}$-arc C starting at a in the direction λ, and not just for a certain one. Again we need a string of lemmas.

17.5 Lemma Let U be a domain, and let K be a compact subset of U. Then there is a real constant M with the property that, whenever f is an analytic function on U taking its values in a strip of width a, $\operatorname{diam} f(K) \leqslant Ma$.

Proof For $z, \zeta \in U$ write $\rho(z,\zeta) = \sup\{|f(z)| : f$ is analytic on U, $|f| < 1$, $f(\zeta) = 0\}$. ρ is continuous on $U \times U$, and

$\rho < 1$. Write $M = (2/\pi)\tanh^{-1}\sup\{\rho(z,\zeta) : z, \zeta \in K\}$. It is sufficient to show the result for the strip $\{z : |\mathrm{Im}\, z| < \frac{\pi}{2}\}$, which has width π. The Gleason metric for this strip is $\dfrac{|\exp s - \exp t|}{|\exp s + \exp \bar{t}|}$ $\geq \tanh \frac{1}{2}|s-t|$. Hence diam $f(K) \leq 2\tanh^{-1}\sup\{\rho(z,\zeta) : z, \zeta \in K\} = M\pi$.

Let V be a domain, and for each sufficiently small positive ϵ let f_ϵ and g_ϵ be complex-valued functions on V. The statement "$f_\epsilon = g_\epsilon + O(\epsilon)$ uniformly on compact subsets of V" means that, whenever K is a compact subset of V, there exist $M \in \underset{\sim}{R}$ and $\epsilon_0 > 0$ such that $|f_\epsilon(z) - g_\epsilon(z)| < M\epsilon$ whenever $z \in K$ and $\epsilon < \epsilon_0$. Other statements of the same type are defined analogously. We shall in future abuse notation by writing, for example, $f = g + O(\epsilon)$ uniformly on compact sets, where it is understood that f and g depend on ϵ. This notation behaves algebraically as one would expect (e.g. if $f_1 = g_1 + O(\epsilon)$ uniformly on compact sets and $f_2 = g_2 + O(\epsilon)$ uniformly on compact sets then $f_1 + f_2 = g_1 + g_2 + O(\epsilon)$ uniformly on compact sets). If f is analytic and $f = O(\epsilon)$ uniformly on compact sets then the Cauchy integral for f' shows that $f' = O(\epsilon)$ uniformly on compact sets. We shall also need the following lemma.

17.6 Lemma Let V be a domain, let $z_0 \in V$, and let f be a function analytic on V such that $f(z_0) > 0$ and $|f| = 1 + O(\epsilon)$ uniformly on compact sets. Then $f = 1 + O(\epsilon)$ uniformly on compact sets.

Proof Let K be a compact subset of V. Choose a domain U with $z_0 \in U$ and $K \subset U \subset \bar{U} \subset V$. By hypothesis there exist $M \in \underset{\sim}{R}$ and $\epsilon_0 > 0$ such that, whenever $\epsilon < \epsilon_0$, $|f| = 1$ with error $M\epsilon$ on $\bar{U}$. We may assume that $M\epsilon_0 \leq \frac{1}{2}$. Taking logarithms, we find that $|\mathrm{Re}(\log f)| \leqslant 2M\epsilon$ on $\overline{U}$ when $\epsilon < \epsilon_0$. By Lemma 17.5 applied to the domain U, there exists a real number N such that $|\log f| \leqslant N\epsilon$ on K when $\epsilon < \epsilon_0$. So $\log f = O(\epsilon)$ uniformly on compact sets. That is, $f = 1 + O(\epsilon)$ uniformly on compact sets.

The next lemma is well-known in the theory of ordinary differential equations.

17.7 <u>Lemma</u> Let σ be a real differentiable function on an interval I, satisfying $|\sigma'| \leqslant K|\sigma| + L$ for some non-negative constants K and L. Then for all $u, v \in I$:

$$|\sigma(v)| \leqslant |\sigma(u)|e^{K|v-u|} + \frac{L}{K}(e^{K|v-u|} - 1).$$

<u>Proof</u> It is sufficient to prove the result when $u < v$ and $\sigma > 0$ on I. The function $e^{-Kx}(\sigma(x) + \frac{L}{K})$ has derivative $e^{-Kx}(\sigma'(x) - K\sigma(x) - L) \leqslant 0$ and so decreases. So $e^{-Kv}(\sigma(v) + \frac{L}{K}) \leqslant e^{-Ku}(\sigma(u) + \frac{L}{K})$, whence the result follows.

17.8 <u>Lemma</u> Let C be a $\frac{1}{2}$-arc starting at 1 in the direction 1. Let $\epsilon > 0$. Let $E_1 = [-1, 1+\epsilon]$ and let E_2 be the union of $[-1,1]$ and a length ϵ of C starting at 1. Then:

$$K_{E_1}(\zeta,\zeta) = K_{E_2}(\zeta,\zeta) + O(\epsilon^2)$$

uniformly on compact subsets of $\underset{\sim}{C} - [-1,1]$.

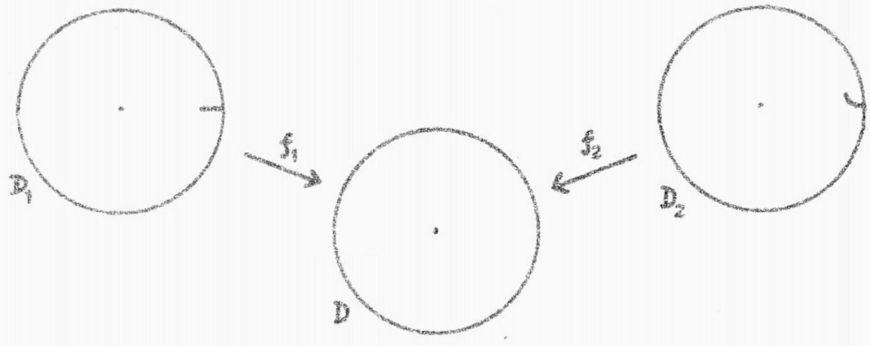
<u>Proof</u> The conformal mapping of $S^2 - [-1,1]$ onto itself which maps 1 to 1 and ζ to ∞ transforms each $\frac{1}{2}$-arc starting at 1 in the direction 1 to a $\frac{1}{2}$-arc starting at 1 in the direction 1, and so the problem is reduced to showing that $\gamma(E_1) = \gamma(E_2) + O(\epsilon^2)$. Now by Corollary 4.3 the analytic capacity of an arc

is at least a quarter of its diameter and at most a quarter of its length. The analytic capacity of E_1 is $\frac{1}{2} + \epsilon/4$, and an easy calculation shows that the diameter and the length of E_2 are both $\frac{1}{2} + \epsilon/4 + O(\epsilon^2)$, whence the result follows.

We need Lemma 17.8 purely as a stepping-stone to the following.

17.9 Lemma Let C be a $\frac{1}{2}$-arc starting at 1 in the direction 1. Let $\epsilon > 0$. Let $E_1 = [-1, 1+\epsilon]$ and let E_2 be the union of $[-1,1]$ and a length ϵ of C starting at 1. Let f be the conformal mapping of $S^2 - E_1$ onto $S^2 - E_2$ such that $f(\infty) = \infty$ and $f'(\infty) > 0$. Then $f(z) = z + O(\epsilon^2)$ uniformly on compact subsets of $\mathbb{C} - [-1,1]$.

Proof Map $S^2 - [-1,1]$ onto the unit disc D by means of the inverse Joukowski transformation $z \to z - \sqrt{z^2-1}$. The image of C is an analytic arc C' starting at 1 in the direction -1. The image of $S^2 - E_1$ is the domain D_1 consisting of the complement, in the unit disc, of a closed interval of length $\sqrt{2}\epsilon^{\frac{1}{2}} + O(\epsilon)$ whose right-hand endpoint is 1. The image of $S^2 - E_2$ is the domain D_2 consisting of the complement, in the unit disc, of the first part of C' of length $\sqrt{2}\epsilon^{\frac{1}{2}} + O(\epsilon)$. For $i = 1$ and $i = 2$ let f_i be the conformal map of D_i onto D satisfying $f_i(0) = 0$ and $f_i'(0) > 0$.



The required result is clearly equivalent to the statement that $f_1 = f_2 + O(\epsilon^2)$ uniformly on compact subsets of D, which we shall show.

Let $R < 1$ and let $|z| \leqslant R$. Let ψ be any conformal mapping of D onto D which takes $f_1(z)$ to $f_2(z)$. Then:

$$(2) \qquad |\psi'(f_1(z))| = \frac{1 - |f_2(z)|^2}{1 - |f_1(z)|^2}$$

$$= 1 \quad \text{with error } M_1(|f_1(z)| - |f_2(z)|)$$

where M_1 depends only on R. Now $f_2^{-1} \circ \psi \circ f_1$ maps D_1 conformally onto D_2 and takes z to z. So the modulus of its derivative at z is $\dfrac{K_1(z,z)}{K_2(z,z)}$, where K_1 and K_2 are the Szegö kernels of D_1 and D_2 respectively. But, by Lemma 17.8 (via the Joukowski transformation), $\dfrac{K_1(z,z)}{K_2(z,z)} = 1 + O(\epsilon^2)$ uniformly for $|z| \leqslant R$. So $|f_1'(z)||\psi'(f_1(z))||f_2'(z)|^{-1} = 1 + O(\epsilon^2)$ uniformly for $|z| \leqslant R$. Using (2), we find:

$$(3) \quad |f_1'(z)| = |f_2'(z)| \quad \text{with error } M_1(|f_1(z)| - |f_2(z)|) + M_2\epsilon^2$$

for sufficiently small ϵ, M_1 and M_2 depending only on R.

Consider, for $i = 1$ or 2, the function $f_i(z)/z$, which is analytic and never zero on D_i, and is positive at 0. If ϵ is sufficiently small, then this has modulus 1 with error $2\epsilon^{\frac{1}{2}}$ near ∂D_i and therefore throughout D_i. Hence, by Lemma 17.6:

$$(4) \qquad f_i(z) = z + O(\epsilon^{\frac{1}{2}}) \quad \text{uniformly on compact sets.}$$

Hence also $f_i' = 1 + O(\epsilon^{\frac{1}{2}})$ uniformly on compact sets.

Let $0 \leqslant \theta < 2\pi$. We shall consider the behaviour of $|f_i(re^{i\theta})|$ for $0 < r \leqslant R$. $\dfrac{d}{dr}|f_i(re^{i\theta})| = |f_i'(re^{i\theta})|\cos\phi$, where ϕ is the difference between $\arg f_i'$ and $\arg f_i - \theta$. But, by (4),

$\arg f_i' = O(\epsilon^{\frac{1}{2}})$ and $\arg f_i - \theta = O(\epsilon^{\frac{1}{2}})$, uniformly for all θ and $r \leq R$. So $\phi = O(\epsilon^{\frac{1}{2}})$. Hence $\cos\phi = 1 + O(\epsilon)$. So $\frac{d}{dr}|f_i(re^{i\theta})| = |f_i'(re^{i\theta})| + O(\epsilon)$. Hence (3) gives:

$$\frac{d}{dr}\{|f_1(re^{i\theta})| - |f_2(re^{i\theta})|\} \leq M_1\{|f_1(re^{i\theta})| - |f_2(re^{i\theta})|\} + M_3\epsilon$$

for all $\epsilon < \epsilon_0$, all θ, and all $r \leq R$, where the constants ϵ_0, M_1 and M_3 depend only on R. Applying Lemma 17.7, with $\sigma(r) = |f_1(re^{i\theta})| - |f_2(re^{i\theta})|$, $u = 0$, and $v = r$, gives:

$$|f_1(re^{i\theta})| - |f_2(re^{i\theta})| \leq \frac{M_3\epsilon}{M_1}(e^{M_1R} - 1),$$

i.e. $|f_1| = |f_2| + O(\epsilon)$ uniformly on compact sets. Applying Lemma 17.6 to f_1/f_2 (which is positive at 0) gives:

(5) $\qquad f_1 = f_2 + O(\epsilon)$ uniformly on compact sets.

The rest of the proof consists of a re-cycling process to improve on (5). We have $f_1(z) = z + O(\epsilon)$ uniformly on compact sets (for if we apply the Joukowski transformation to the domain and range of f_1, then we see that this just says that the conformal mapping of $S^2 - [-1,1+\epsilon]$ onto $S^2 - [-1,1]$, fixing ∞, and having positive derivative at ∞, has the form $z \to z + O(\epsilon)$ uniformly on compact subsets of $\underset{\sim}{C} - [-1,1]$, which is obvious). So, by (5), $f_2(z) = z + O(\epsilon)$ uniformly on compact sets. We can now use these improvements on (4) to deduce, by exactly the method of the last paragraph, that $f_1 = f_2 + O(\epsilon^2)$, as required.

There are two easy generalisations of Lemma 17.9. First, let C_1 and C_2 be $\frac{1}{2}$-arcs starting at 1 in the direction 1. For $i = 1$ and $i = 2$ and for $\epsilon > 0$ let E_i be the union of $[-1,1]$ and a length ϵ of C_i starting at 1. Then the conformal map of $S^2 - E_2$ onto $S^2 - E_1$, fixing ∞ and having positive derivative there, is of the form $z \to z + O(\epsilon^2)$ uniformly on compact subsets of $\underset{\sim}{C} - [-1,1]$. The proof consists of two applications of Lemma 17.9. Secondly, let $a, \lambda \in \underset{\sim}{C}$, $|\lambda| = 1$. Let E be a compact connected set consisting of the union of an analytic arc starting at a in the direction $-\lambda$ and a compact set not containing a. Suppose that $a \in \partial\Omega(E)$. Let C_1 and C_2 be $\frac{1}{2}$-arcs starting at a in the direction λ. For $i = 1$ and $i = 2$ and for $\epsilon > 0$ let E_i be the union of E and a length ϵ of C_i starting at a. Let $\zeta \in \Omega(E)$. Then the conformal map of $\Omega(E_2)$ onto $\Omega(E_1)$, fixing ζ and having positive derivative there, is of the form $z \to z + O(\epsilon^2)$ uniformly on compact subsets of $\Omega(E) - \{\infty\}$. To prove this, map $\Omega(E)$ conformally onto $S^2 - [-1,1]$ in such a way that ζ is mapped to ∞ and a is mapped to 1, and apply the last result.

17.10 Lemma Let A^ρ be the annulus $\{z : 1 < |z| < \rho\}$, and let K^ρ and L^ρ be its Szegö kernel and co-kernel respectively. Then $K^\rho(z,\zeta)$ and $L^\rho(z,\zeta)$ are continuously differentiable functions of (z,ζ,ρ) inside their domains of definition.

Proof The functions $u_n(z) = \dfrac{1}{\sqrt{2\pi}}\dfrac{z^n}{\sqrt{1+\rho^{2n+1}}}$ $(n = 0, \pm 1, \pm 2, \ldots\ldots)$ form an orthonormal basis for $H^2(A^\rho)$. Hence $K^\rho(z,\zeta) = \dfrac{1}{2\pi}\sum_{-\infty}^{\infty}\dfrac{z^n\bar{\zeta}^n}{1+\rho^{2n+1}}$. Hence also, by (4) of §8:

$$L^\rho(z,\zeta) = \frac{1}{2\pi(z-\zeta)} + \frac{1}{2\pi}\int_{\partial A^\rho}\frac{K^\rho(\zeta,\eta)}{\eta - z}|d\eta|$$
$$= \frac{1}{2\pi(z-\zeta)} + \frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{(z^{2n+1} - \zeta^{2n+1})}{(z\zeta)^{n+1}(1+\rho^{2n+1})}.$$

Elementary calculus shows that these expressions are continuously differentiable.

A <u>ring domain</u> is a doubly-connected domain in S^2 with no isolated boundary points. It is well known (see, for example, [4] p. 209) that if U is a ring domain then there is a unique number $\rho > 1$, the <u>Riemann modulus</u> of U, such that U can be mapped conformally onto the annulus $\{z : 1 < |z| < \rho\}$. The proof of that fact in [4] shows also that if U and V are ring domains with one common boundary component and $U \subset V$ then the Riemann modulus of U is not greater than the Riemann modulus of V.

17.11 <u>Lemma</u> Let $\rho > 1$. Let U be the annulus $\{z : 1 < |z| < \rho\}$. Let Q be a compact subset of U. Then there exist positive numbers M and ϵ_0 such that whenever $0 < \epsilon < \epsilon_0$ and U_1 is a ring domain with $\{z : 1 < |z| < \rho - \epsilon\} \subset U_1 \subset \{z : 1 < |z| < \rho + \epsilon\}$, then:

$$|K_1(z,\zeta) - K(z,\zeta)| \leq M\epsilon \qquad (z, \zeta \in Q)$$

$$|L_1(z,\zeta) - L(z,\zeta)| \leq M\epsilon \quad (z, \zeta \in Q,\ z \neq \zeta)$$

where K and K_1 are the Szegö kernels of U and U_1 respectively and L and L_1 are the corresponding co-kernels.

<u>Proof</u> We shall show the result for K: L is similar. M_1, M_2, etc., will denote constants depending only on ρ and Q. Choose $\epsilon_1 < d(Q, \partial U)$. Lemma 17.10 shows that $|K^s(z,\zeta) - K(z,\zeta)| \leq M_1\epsilon$ whenever $\epsilon < \epsilon_1$, $\rho - \epsilon \leq s \leq \rho + \epsilon$, and $z, \zeta \in Q$. (Here K^s is the Szegö kernel of $\{z : 1 < |z| < s\}$.) Now let $0 < \epsilon < \epsilon_1$, and let U_1 be any ring domain with $\{z : 1 < |z| < \rho - \epsilon\} \subset U_1 \subset \{z : 1 < |z| < \rho + \epsilon\}$. Let ϕ be a conformal mapping of U_1 onto an annulus $\{z : 1 < |z| < s\}$. By the remarks in the last paragraph on the monotonicity of the Riemann modulus, $\rho - \epsilon \leq s \leq \rho + \epsilon$. The function

$\phi(z)/z$, defined on U_1, has modulus between $1 - \epsilon/\rho$ and $1 + \epsilon/\rho$ on U_1. (Consider its behaviour near the boundary of U_1 and use the maximum modulus theorem.) So by Lemma 17.6, replacing ϕ if necessary by $\lambda\phi$ for some $\lambda \in \underset{\sim}{C}$ with $|\lambda| = 1$, $|\phi(z) - z| \leqslant M_2\epsilon$ and $|\phi'(z) - 1| \leqslant M_3\epsilon$ when $\epsilon < \epsilon_0$ and $z \in Q$. Here ϵ_0 depends only on ρ and Q. Now let $z, \zeta \in Q$.

$$\begin{aligned} K_1(z,\zeta) &= \phi'(z)^{\frac{1}{2}}\, \overline{\phi'(\zeta)^{\frac{1}{2}}}\, K^S(\phi(z),\phi(\zeta)) \\ &= K^S(z,\zeta) \quad \text{with error } M_4\epsilon, \text{ for sufficiently small } \epsilon \\ &= K(z,\zeta) \quad \text{with error } (M_1 + M_4)\epsilon, \text{ for sufficiently small } \epsilon. \end{aligned}$$

The following is an easy generalisation of Lemma 17.11.

17.12 <u>Lemma</u> Let U be a ring domain in S^2, one of the components of whose boundary is an analytic Jordan curve Γ. For $\epsilon > 0$ define $U^{\epsilon} = \{z : z \in U \text{ or } d(z,\Gamma) < \epsilon\}$ and $U^{-\epsilon} = \{z : z \in U \text{ and } d(z,\Gamma) > \epsilon\}$. Let Q be a compact subset of U. Then there exist positive numbers M and ϵ_0 such that whenever $0 < \epsilon < \epsilon_0$ and U_1 is a ring domain with $U^{-\epsilon} \subset U_1 \subset U^{\epsilon}$ then:

$$|K_1(z,\zeta) - K(z,\zeta)| \leqslant M\epsilon \qquad (z, \zeta \in Q)$$
$$|L_1(z,\zeta) - L(z,\zeta)| \leqslant M\epsilon \quad (z, \zeta \in Q,\ z \neq \zeta)$$

where K and K_1 are the Szegö kernels of U and U_1 respectively and L and L_1 are the corresponding co-kernels.

<u>Proof</u> Map U conformally onto an annulus $\{z : 1 < |z| < \rho\}$ so that Γ corresponds to the circle $\{z : |z| = \rho\}$. The mapping continues analytically across Γ and the result follows easily from Lemma 17.11.

17.13 <u>Lemma</u> Let E be a compact subset of $\underset{\sim}{C}$, and consist either of one component, not a singleton, or of two separated components, neither one a singleton. Let $a \in E$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Suppose that E is the union of an analytic arc Γ starting at a in the direction $-\lambda$ and a compact set not containing a. Let C_1 and C_2 be $\frac{1}{2}$- arcs starting at a in the direction λ. Let $\epsilon > 0$. For $i = 1$ and $i = 2$ let E_i be the union of E and a length ϵ of C_i starting at a. Then:

$$K_{E_1} = K_{E_2} + O(\epsilon^2) \quad \text{uniformly on compact subsets of } \Omega(E) \times \Omega(E)$$
$$L_{E_1} = L_{E_2} + O(\epsilon^2) \quad \text{uniformly on compact subsets of } \Omega(E) \times \Omega(E).$$

<u>Proof</u> We may suppose that $a \in \partial\Omega(E)$, as otherwise the result is trivial. We prove the result for K: L is similar.

Case 1: E is connected. Let ϕ be the conformal map of $\Omega(E_2)$ onto $\Omega(E_1)$ such that $\phi(\infty) = \infty$ and $\phi'(\infty) > 0$. By the remarks after Lemma 17.9, $\phi(z) = z + O(\epsilon^2)$ uniformly on compact subsets of $\mathbb{C}$. So:

$$\begin{aligned} K_{E_2}(\zeta_1, \zeta_2) &= \phi'(\zeta_1)^{\frac{1}{2}}\, \overline{\phi'(\zeta_2)^{\frac{1}{2}}}\, K_{E_1}(\phi(\zeta_1), \phi(\zeta_2)) \\ &= (1 + O(\epsilon^2))(1 + O(\epsilon^2))(K_{E_1}(\zeta_1, \zeta_2) + O(\epsilon^2)) \\ &= K_{E_1}(\zeta_1, \zeta_2) + O(\epsilon^2) \end{aligned}$$

uniformly on compact sets.

Case 2: E has two components. Denote by E' and E'' the components of E containing a and not containing a respectively. If we apply any conformal mapping of $\Omega(E'')$ then the problem is unchanged: so we can assume that E'' is an analytic Jordan curve S. Denote by E_1' and E_2' the components of E_1 and E_2 respectively containing a. Now let ϕ be the conformal map of $\Omega(E_2')$ onto $\Omega(E_1')$ such that $\phi(\infty) = \infty$ and $\phi'(\infty) > 0$. $\phi(z) = z + O(\epsilon^2)$

uniformly on compact subsets of $\Omega(E') - \{\infty\}$. Hence there is a constant M such that for all sufficiently small $\epsilon > 0$, $|\phi(z) - z| < M\epsilon^2$ $(z \in S)$. In other words, for sufficiently small ϵ, ϕ maps $\Omega(E_2)$ onto a domain V contained in $\{z : z \in \Omega(E_1)$ or $d(z,S) < M\epsilon^2\}$ and containing $\{z : z \in \Omega(E_1)$ and $d(z,S) > M\epsilon^2\}$. So by Lemma 17.12 $K_V = K_{E_1} + 0(\epsilon^2)$ uniformly on compact sets, where K_V is the Szegö kernel of V. Hence $K_{E_2}(\zeta_1,\zeta_2) = \phi'(\zeta_1)^{\frac{1}{2}}\, \overline{\phi'(\zeta_2)^{\frac{1}{2}}}\, K_V(\phi(\zeta_1),\phi(\zeta_2)) = K_V(\zeta_1,\zeta_2) + 0(\epsilon^2) = K_{E_1}(\zeta_1,\zeta_2) + 0(\epsilon^2)$.

Lemma 17.4 and Lemma 17.13 now combine to give the following.

17.14 <u>Theorem</u> Let E be a compact subset of $\underset{\sim}{C}$, and consist either of one component, not a singleton, or of two separated components, neither one a singleton. Let $a \in E$, $\lambda \in \underset{\sim}{C}$, $|\lambda| = 1$. Suppose that E is the union of an analytic arc Γ starting at a in the direction $-\lambda$ and a compact set not containing a. Let ζ_1 and ζ_2 be finite points of $\Omega(E)$. Let α be a quarter of the limit of $\mathrm{Pert}(K_E(\zeta_1,\zeta_2);z)$ as $z \to a$ along any curve entering a along the line $\{a + t\lambda : t \geq 0\}$. Let C be a $\frac{1}{2}$-arc starting at a in the direction λ. Let E_t be the union of E and a length t of C starting at a. Then:

$$K_{E_t}(\zeta_1,\zeta_2) = K_E(\zeta_1,\zeta_2) + \alpha t + 0(t^2).$$

If in addition $\zeta_1 \neq \zeta_2$ then the same result holds with K replaced throughout by L.

<u>Proof</u> Lemma 17.4 says that there is such a $\frac{1}{2}$-arc C, and the result follows for all other $\frac{1}{2}$-arcs by Lemma 17.13.

In our applications of Theorem 17.14, C will in fact be the continuation of Γ. The result of Theorem 17.14 applies not only to K_E and L_E, but also, by elementary calculus, to quantities obtained from these by algebraic operations, and in particular to $\text{Pert}(K_E(\zeta,\zeta);\eta) = |L_E(\eta,\zeta)|^2 - |K_E(\eta,\zeta)|^2$. We shall want to apply it to $\gamma(E)$ and to $a_E(\eta)$; its applicability to these follows, by a linear fractional transformation, from its applicability to $K_E(\zeta,\zeta)$ and to $\text{Pert}(K_E(\zeta,\zeta);\eta)$.

§18 Applications to Subadditivity Problems

The usefulness of Theorem 17.14 is that it allows us to "build up" sets consisting of a union of analytic arcs by gradually extending these arcs. We shall need the following lemma.

18.1 Lemma Let E be a compact subset of $\mathbb{C}$. Let $a \in E$. Suppose that E has finitely many components, and that some conformal map of the complement in S^2 of the component of E containing a onto D extends to be continuous at a. Let $\{E_n\}$ be a sequence of compact subsets of E with the property that, given any $r > 0$, $E - D(a;r) = E_n - D(a;r)$ for all sufficiently large n. Then $\gamma(E_n) \to \gamma(E)$, and $a_{E_n}(z) \to a_E(z)$ for all $z \in \Omega(E) - \{\infty\}$.

Proof Let $\delta > 0$. The assumptions on E show that f_E extends to be continuous at a (compare Corollary 7.2). So by Corollary 5.3 f_E can be uniformly approximated by functions which are analytic on $\Omega(E)$ and at a. Hence we can find a function f, analytic on $\Omega(E)$ and at a, bounded by 1, and satisfying $|f'(\infty)| \geq \gamma(E) - \delta$. For all sufficiently large n, f is admissible for E_n, and so $\gamma(E_n) \geq \gamma(E) - \delta$. This holds for all $\delta > 0$ and so lim inf $\gamma(E_n) \geq \gamma(E)$. But $\gamma(E_n) \leq \gamma(E)$ for each n: so $\gamma(E_n) \to \gamma(E)$.

Let $z \in \Omega(E) - \{\infty\}$. By Theorem 11.3, there exist positive numbers ϵ_0 and k and a positive integer n_0 such that whenever $n > n_0$ and $\epsilon < \epsilon_0$:

$$(1) \qquad |\gamma(E_n \cup \bar{D}(z;\epsilon)) - \gamma(E_n) - \epsilon a_{E_n}(z)| < k\epsilon^2.$$

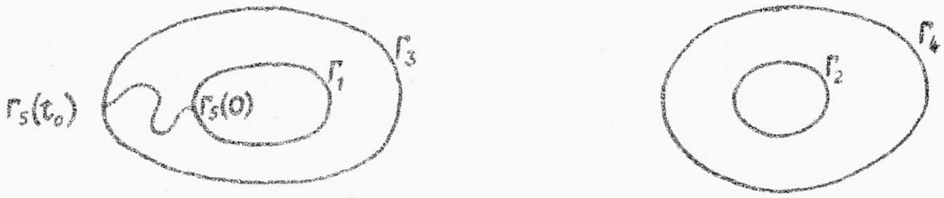
Hence the sequence $\{\epsilon a_{E_n}(z)\}_{n>n_0}$, considered as an element of the Banach space of all bounded sequences with the supremum norm, is within a distance $k\epsilon^2$ of the sequence $\{\gamma(E_n \cup \bar{D}(z;\epsilon)) - \gamma(E_n)\}$. We have seen that $\gamma(E_n) \to \gamma(E)$; similarly $\gamma(E_n \cup \bar{D}(z;\epsilon)) \to \gamma(E \cup \bar{D}(z;\epsilon))$. So $\{a_{E_n}(z)\}_{n>n_0}$ is, for all ϵ, within a distance $k\epsilon$ of the closed subspace c of all sequences which converge, and is therefore itself in c. Letting $n \to \infty$ in (1) shows that its limit is $a_E(z)$ as required.

We can now prove a string of results on the slope function and analytic capacity.

18.2 Theorem Let E and F be compact subsets of $\mathbb{C}$, each having at most two components. Suppose $E \subset F$. Then $a_F(z) \leq a_E(z)$ $(z \in \Omega(F) - \{\infty\})$.

Proof Let $z \in \Omega(F) - \{\infty\}$. Since the slope functions of a decreasing sequence of compact sets converge to the slope function of the intersection, we can assume that E and F are each the union of one or two separated analytic Jordan curves and their insides, and that $E \subset \text{int } F$. We shall prove the result in the case when $\partial\Omega(E)$ consists of two analytic Jordan curves Γ_1 and Γ_2 and $\partial\Omega(F)$ consists of two analytic Jordan curves Γ_3 and Γ_4, with Γ_1

inside Γ_3 and Γ_2 inside Γ_4: the technique is the same in each of the other four possible configurations.



Choose any analytic arc $\Gamma_5(t)$ $(0 \leq t \leq t_0)$, parametrised by length, starting on Γ_1, ending on Γ_3, and lying (apart from its endpoints) in the ring domain between Γ_1 and Γ_3. For $0 \leq t \leq t_0$ write $E_t = E \cup \Gamma_5[0,t]$. By the remarks following Theorem 17.14, $\frac{d^+}{dt}\{a_{E_t}(z)\} = \frac{1}{4}\lim_{s\to t+} \mathrm{Pert}(a_{E_t}(z);\Gamma_5(s))$. But $\mathrm{Pert}(a_{E_t}(z);\Gamma_5(s)) = b_{E_t}(z,\Gamma_5(s)) \leq 0$ by Corollary 16.8 since E_t has only two components. So $\frac{d^+}{dt}\{a_{E_t}(z)\} \leq 0$. Also $a_{E_t}(z)$ is a continuous function of t for $0 \leq t \leq t_0$ by Lemma 18.1. So $a_{E_{t_0}}(z) \leq a_E(z)$. Now let $\Gamma_3(t)$ $(0 \leq t \leq t_1)$ be a parametrisation of Γ_3 by length, starting and finishing at $\Gamma_5(t_0)$. Write $E_t' = E_{t_0} \cup \Gamma_3[0,t]$. As before, $\frac{d^+}{dt}\{a_{E_t'}(z)\} \leq 0$ and $a_{E_t'}(z)$ is a continuous function of t. So $a_{E_{t_1}'}(z) \leq a_{E_0'}(z) = a_{E_{t_0}}(z) \leq a_E(z)$; that is, $a_{E'}(z) \leq a_E(z)$, where E' is the union of Γ_2 and Γ_3 and their insides. The same technique, applied to Γ_2 and Γ_4, gives $a_F(z) \leq a_{E'}(z)$. The result follows.

18.3 <u>Corollary</u> Let F be a compact subset of $\mathbb{C}$ with at most two components. Then $a_F \leq 1$.

<u>Proof</u> Choose $\zeta \in F$ and write $E = \{\zeta\}$: so $a_E = 1$.

18.4 <u>Theorem</u> Let E_1 and E_2 be compact connected subsets of $\mathbb{C}$. Then $\gamma(E_1 \cup E_2) \leq \gamma(E_1) + \gamma(E_2)$.

Proof We may assume that for $i = 1$ and $i = 2$ E_i consists of an analytic Jordan curve Γ_i and its inside.

Case 1: $E_1 \cap E_2 = \phi$. Let $\Gamma_1(t)$ $(0 \leq t \leq t_0)$ be a parametrisation of Γ_1 by length. For $0 \leq t \leq t_0$ write $K_t = \Gamma_1[0,t]$ and $K_t' = \Gamma_1[0,t] \cup \Gamma_2$. Let $0 < t < t_0$. Theorem 17.14 says that:

$$\frac{d^+}{dt}(\gamma(K_t)) = \frac{1}{4}\lim_{s \to t+} a_{K_t}(\Gamma_1(s))$$
$$\frac{d^+}{dt}(\gamma(K_t')) = \frac{1}{4}\lim_{s \to t+} a_{K_t'}(\Gamma_1(s)).$$

Subtracting, we have:

$$\begin{aligned} \frac{d^+}{dt}(\gamma(K_t') - \gamma(K_t)) &= \frac{1}{4}\lim_{s \to t+} \{a_{K_t'}(s) - a_{K_t}(s)\} \\ &\leq 0 \end{aligned} \tag{2}$$

by Theorem 18.2. Also $\gamma(K_t') - \gamma(K_t)$ is a continuous function of t, by Lemma 18.1. Hence:

$$\gamma(K_{t_0}') - \gamma(K_{t_0}) \leq \gamma(K_0') - \gamma(K_0) \tag{3}$$

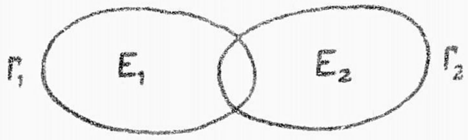
i.e. $\quad \gamma(E_1 \cup E_2) - \gamma(E_1) \leq \gamma(E_2) - 0$

i.e. $\quad \gamma(E_1 \cup E_2) \leq \gamma(E_1) + \gamma(E_2)$.

Case 2: $E_1 \cap E_2 \neq \phi$. The proof is basically the same, but the details must be modified. Since Γ_1 and Γ_2 are analytic curves, we know that for each t, $0 \leq t < t_0$, <u>either</u> for all s larger than t and sufficiently near t, $\Gamma_1(s) \in \Omega(K_t')$, <u>or</u> for all s larger than t and sufficiently near t, $\Gamma_1(s) \notin \Omega(K_t')$. In the first case, the proof of (2) still holds, unless $\Gamma_1(t) \in \Gamma_2$. In the second case, $\gamma(K_s') = \gamma(K_t')$ for all s larger than t and sufficiently near t, so that $\frac{d^+}{dt}(\gamma(K_t') - \gamma(K_t)) = -\frac{d^+}{dt}(\gamma(K_t)) \leq 0$. Thus (2) holds for all t except possibly those (finitely many) t for which $\Gamma_1(t) \in \Gamma_2$. (3) follows as before.

In the case when $E_1 \cap E_2 \neq \phi$, Theorem 18.4 is in fact a known result. For, in that case, E_1, E_2 and $E_1 \cup E_2$ are all connected, so that the result can be re-written as $\mathrm{cap}(E_1 \cup E_2) \leq \mathrm{cap}(E_1) + \mathrm{cap}(E_2)$ by Proposition 4.1. Pommerenke has shown ([19], theorem 2) that if E_1 and E_2 are compact plane sets and $E_1 \cup E_2$ is connected then $\mathrm{cap}(E_1 \cup E_2) \leq \mathrm{cap}(E_1) + \mathrm{cap}(E_2)$.

The technique of the last proof can also be used to give strong subadditivity in certain cases. Consider, for example, the following configuration:



Let $\Gamma_1(t)$ $(0 \leq t \leq t_0)$ be a parametrisation of that part of Γ_1 not inside Γ_2. Write $K_t = (E_1 \cap E_2) \cup \Gamma_1[0,t]$ and $K_t' = E_2 \cup \Gamma_1[0,t]$. Then, as before:

$$\gamma(K_{t_0}') - \gamma(K_{t_0}) \leq \gamma(K_0') - \gamma(K_0)$$

i.e. $$\gamma(E_1 \cup E_2) - \gamma(E_1) \leq \gamma(E_2) - \gamma(E_1 \cap E_2)$$

i.e. $$\gamma(E_1 \cup E_2) \leq \gamma(E_1) + \gamma(E_2) - \gamma(E_1 \cap E_2).$$

The following Theorem 18.5 seems to be the most general result that can be obtained by that method. We omit the proof because the technique is exactly as before and the details would be clumsy if written out. Theorem 18.5 of course contains Theorem 18.4.

18.5 <u>Theorem</u> Let E_1 and E_2 be compact subsets of $\mathbb{C}$. Suppose that each of E_1, E_2 and $E_1 \cup E_2$ has at most two components. Then $\gamma(E_1 \cup E_2) \leq \gamma(E_1) + \gamma(E_2) - \gamma(F)$ where F is any subset of $E_1 \cap E_2$ with at most two components. In particular, $\gamma(E_1 \cup E_2) \leq \gamma(E_1) + \gamma(E_2)$.

§19 Unsolved Problems

The purpose of this brief final section is to gather into one place the open questions which have arisen in this thesis or which arise naturally out of other material in it.

(1) Is γ strongly subadditive: i.e. $\gamma(E_1 \cup E_2) \leq \gamma(E_1) + \gamma(E_2) - \gamma(E_1 \cap E_2)$ for all compact plane sets E_1 and E_2?

(2) Is γ subadditive: i.e. $\gamma(E_1 \cup E_2) \leq \gamma(E_1) + \gamma(E_2)$ for all compact plane sets E_1 and E_2?

(3) Is a_E a decreasing function of E for arbitrary compact plane sets E?

(4) Is $b_E(z,\zeta) \leq 0$ for all compact plane sets E?

(5) Is $b_E(z,\zeta,a) \leq 0$ for all compact sets $E \subset \underset{\sim}{R}$?

(6) Is $a_E \leq 1$ for all compact plane sets E?

(7) Is there a universal constant M such that $a_E \leq M$ for all compact plane sets E?

The only relations among the above seven questions that I know of are the trivial ones: (1) $\Rightarrow$(2) $\Rightarrow$(6) $\Rightarrow$(7), (3) $\Rightarrow$(6) $\Rightarrow$(7), (1) $\Rightarrow$(3) $\Rightarrow$(4) $\Rightarrow$(5). It ought to be possible to answer (5) by a computational proof or a counterexample: but it is not clear how to set about looking for affirmative answers to any of the others.

The obvious question to ask in connection with §17 is:

(8) Does Theorem 17.14 remain true for sets E with arbitrarily (finitely) many components?

I should say that the answer to (8) is almost certainly affirmative: I feel that this is a natural result, and that it is only limitations in technique that have restricted us to the case of at most two components. If Theorem 17.14 does generalise in that way, then the method of §18 shows that (1), (3) and (4) are equivalent: indeed that (4) is nothing other than a differentiated form of (1).

I feel that confidence in the truth of (4) itself is not so justified. It may well be true: but the only grounds we have for conjecturing its truth is that it is true when $E \in J$, and we have already seen, in §15, the danger in basing a conjecture on such a special case as that.

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LIST OF NOTATIONS

The following list contains only those symbols which are used as standard throughout the thesis.